Cosmological Perturbation Theory
Derived and applied to a Robertson-Walker universe

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Abstract

Common literature about cosmological perturbation theory starts with the harmonic decomposition of perturbations and goes a long way deriving and explaining the perturbation equations.

In a Robertson-Walker universe, a different approach is to solve Cartan's equations of structure in a 3+1 split of spacetime and find gauge invariant perturbation equations through straight calculation. The applications shown include finding the sources for density fluctuations as perturbations of entropy and anisotropic pressure.

Contents

1 Introduction

The General Theory of Relativity describes our universe based on Einstein's famous equations. But only few physically interesting exact solutions of Einstein's equations known, and these solutions, such as the Robertson-Walker solution or the Schwarzschild solution, leave many questions unanswered:

Where do galaxies come from? The evolution of density fluctuations might be able to explain the existence and formation of these structures, so the question is: How does a small inhomogeneity in a nearly Robertson-Walker universe evolve? Do primordial density fluctuations have the necessary properties to create the universe we see today if they are expanded?

To answer these questions, one must find equations that describe how physical quantities behave under slight displacements in spacetime, in other words, *how a perturbed system evolves with time.*

In this paper, we will derive the perturbed Einstein equations for a Robertson-Walker universe, which is homogeneous and isotropic. It describes what our universe looks like on large scales.[1]

Once we know the perturbed Einstein equations, we use the conservation of the stress-energy tensor to *find out how density contrast and velocities evolve and what generates density fluctuations: The perturbations of entropy and anisotropic pressure.*

Appendix A shows how to manually calculate the exact solution of Einstein's equations for the Robertson-Walker metric.

Since existing literature on cosmological perturbation theory is not only rare but also fairly complicated, we chose an approach to the subject which simplifies calculations and sticks to the physics.

Demystifying the subject, we have included step by step calculations rather than just stating the results. Where we left something without explanation or proof, we referenced literature or assumed that the reader will know how to deal with it out of context.

From this point of view, the paper on hand serves as an introduction to cosmological perturbation theory which enables the reader to understand where the perturbation equations come from and dive into the complete treatment of cosmological perturbation theory, as compiled by Kodama and Sasaki in [1].

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[1] A complete description of the standard cosmological model is given in [4], chapter 1.

2 Preliminaries

In the the late 1920s, Cartan figured he could package the 21 components of the Riemann tensor $R_{\mu\nu\alpha\beta}$ into only six curvature forms $\Omega^i{}_j$, using amongst other things the exterior derivative d to abbreviate calculations elegantly. One major advantage of this *cartan formalism* is that often, terms cancel out before they have to be calculated.[2]

In this section, we use this approach combined with the idea of splitting a spacetime manifold[3] into space parts and time parts, which simplifies calculations even more, to arrive at expressions for Ricci and Einstein tensor that will facilitate the derivation of the perturbation equations for a homogeneous, isotropic universe.

2.1 3+1 Split Of Spacetime

A spacetime manifold (M, g) can be decomposed with a diffeomorphism

$$\Phi : M \longrightarrow \Sigma \times I,$$

where Σ is regarded as the space-part and $I \subset \mathbf{R}$ as the time-part of M. More precisely,

$$\Sigma_t := \Phi^{-1}(\Sigma \times \{t\}) \tag{1}$$

are three-dimensional manifolds, and

$$\gamma_x := \Phi^{-1}(\{x\} \times I) \tag{2}$$

are timelike curves. Imagine M to consist of Σ_t-slices pierced with γ_x-curves. It is very natural, of course, to call this decomposition the *3+1 split of spacetime*.

Imagine the vector field ∂_t, pointing in the direction of γ_x at every point $p \in \Sigma_t$. This vector field can be decomposed into a part normal to Σ_t and a part that lies in Σ_t:

$$\partial_t := \alpha \cdot n + \bar{\beta}. \tag{3}$$

We call α the *lapse function*, $n \perp \Sigma_t$ is the normal vector to Σ_t, $\bar{\beta} \in \Sigma_t$ is called the *shift vector*. This is the first application of the 3+1 split of spacetime. We are going to calculate many expressions applying the same principle. The metric g, for example, can be splitted like

$$g = \eta_{\mu\nu}\theta^\mu \otimes \theta^\nu = -\theta^0 \otimes \theta^0 + \delta_{ij}\theta^i \otimes \theta^j.$$

Note that in all following calculations, one must pay attention whether Latin indices do not run from 0 to 3, but only from 1 to 3, since the time-part of spacetime has been decomposed from the space-part.

[2] A good account of this idea and its implications is given in [2], chapter 14.5.

[3] Readers who are not familiar with fundamental notions of differential geometry (manifolds, vector fields, curves, connections, and the like) may refer to chapters one to three of [6], Robert Wald's famous treatment of general relativity.

Consider vector fields $\{e_\mu\}$ and one-forms $\{\theta^\mu\}$, orthonormal bases of the tangent space $T(\Sigma)$ or the cotangent space $T^*(\Sigma)$, respectively, defined as follows:

$$\begin{aligned} e_0 &:= n = \frac{1}{\alpha}\left(\partial_t - \bar{\beta}\right) \qquad \text{using (3)} && (4)\\ \theta^0 &:= \alpha \cdot dt && (5)\\ e_i &:= \bar{e}_i && (6)\\ \theta^j &:= \bar{\theta}^j + \beta^j dt. && (7)\end{aligned}$$

The bar over certain characters is used to mark them as elements of the tangent space $T(\Sigma)$, or $T^*(\Sigma)$, respectively.

With direct calculation one verifies that

$$\bar{\theta}^j\left(\bar{e}_i\right) = \delta^i{}_j,$$

a fact we will use quite often.

With regard to a coordinate basis,

$$\begin{aligned} g(\partial_t, \partial_t) &= -\theta^0\left(\partial_t\right)\theta^0\left(\partial_t\right) + \delta_{ij}\theta^i\left(\partial_t\right)\theta^j\left(\partial_t\right)\\ &= -\alpha^2 + \delta_{ij}\beta^i\beta^j = -\alpha^2 + \beta^i\beta_i && (8)\\ g(\partial_t, \bar{e}_i) &= -\theta^0\left(\partial_t\right)\theta^0\left(\bar{e}_i\right) + \delta_{ij}\theta^i\left(\partial_t\right)\theta^j\left(\bar{e}_i\right)\\ &= -\alpha\cdot\alpha dt\left(\bar{e}_i\right) + \delta_{ij}\left(\beta^i\left[\bar{\theta}^j(\bar{e}_i) + \beta^j dt(\bar{e}_i)\right]\right)\\ &= \beta_i. && (9)\end{aligned}$$

Note that we made use of the fact that in our 3+1 split, $e_0 \perp \bar{e}_i \quad \forall i$: Space and time stand normal to each other.

With (8) and (9), our metric becomes

$$\begin{aligned} g &= \left[-\alpha^2 + \beta^i\beta_i\right]dt^2 + 2\beta_i dt dx^i + \bar{g}_{ij}dx^i dx^j\\ &= -\alpha^2 + \bar{g}_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right). && (10)\end{aligned}$$

2.2 Cartan Formalism

The aim of this section is to calculate the Einstein tensor in terms of lapse function α, shift vector $\bar{\beta}$ and metric $\bar{g}$. We will use this result afterwards for our formulation of cosmological perturbation theory, which will thus become much easier to handle.

First, we note that curvature defines[4] differential forms $\Omega^i{}_j$, called *curvature forms*, like

$$R(X,Y)e_j := \Omega^i{}_j(X,Y)e_i.$$

In components,

[4] According to [5], p. 68.

$$R_{\mu\nu} = \Omega^{\sigma}{}_{\mu}(e_\sigma, e_\nu). \tag{11}$$

As will become clear using (19), the definition of $\wedge$ and the properties of ω_{ij},

$$\Omega_{\mu\nu} = -\Omega_{\nu\mu}.$$

It follows that

$$\begin{aligned} \Omega^0{}_0 &= \eta^{0\lambda}\Omega_{\lambda 0} = -\Omega_{00} = \Omega_{00} = 0 \\ \Omega^0{}_i &= -\Omega_{0i} = \Omega_{i0} = \Omega^i{}_0. \end{aligned}$$

Applying our 3+1 split, we get the components of the Ricci tensor with time, time-space and space coordinates expressed in curvature forms:

$$\begin{aligned} R_{00} &= \Omega^i{}_0(e_i, e_0) = \Omega^0{}_i(e_i, e_0) \\ R_{0i} &= \Omega^j{}_0(e_j, e_i) = \Omega^0{}_j(e_j, e_i) \\ R_{ij} &= \Omega^0{}_i(e_0, e_j) + \Omega^k{}_i(e_k, e_i). \end{aligned}$$

Let us look at how far we already get calculating the Einstein tensor. The G_{00} component is

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}\eta_{00}R = R_{00} - \frac{1}{2}\eta_{00}R^\lambda{}_\lambda \\ &= R_{00} + \frac{1}{2}\left(R^0{}_0 + R^i{}_i\right) = R_{00} + \frac{1}{2}\left(\eta^{0\lambda}R_{\lambda 0} + \eta^{i\lambda}R_{\lambda i}\right) \\ &= R_{00} - \frac{1}{2}R_{00} + \frac{1}{2}R_{ii} = \frac{1}{2}\left(R_{00} + R_{ii}\right) \\ &= \frac{1}{2}\left(\Omega^0{}_i(e_i, e_0) + \Omega^0{}_i(e_0, e_i) + \Omega^k{}_i(e_k, e_i)\right) \\ &= \frac{1}{2}\Omega^k{}_i(e_k, e_i), \end{aligned} \tag{12}$$

and since e_0 does not appear in this or the R_{0i} equation anymore, and since according to (4) $e_i = \bar{e}_i$, G_{00} and R_{0i} depend only on the curvature forms that are defined on the tangent space $T(\Sigma)$:

$$R_{0i} = \Omega^0{}_j(\bar{e}_j, \bar{e}_i) \tag{13}$$

$$G_{00} = \frac{1}{2}\Omega^k{}_i(\bar{e}_k, \bar{e}_i). \tag{14}$$

2.2.1 Cartan's Equations Of Structure

The *equations of structure* give us a link between the curvature forms and the so-called *connection one-forms*, and with a Cartan lemma we can replace the connection one-forms with functions that have a direct expression in the metric and the lapse function. This way, at the end we will have the Ricci tensor and therefore also Einstein's equations in terms of the metric, which is what we are looking for.

We start with the *first equation of structure*[5] for the connection one-forms ${\omega^\mu}_\nu$ on a manifold (M, g), in an orthonormal basis[6]:

$$\begin{aligned} d\theta^0 + {\omega^0}_i \wedge \theta^i &= 0 & (15) \\ d\theta^i + {\omega^i}_0 \wedge \theta^0 + {\omega^i}_j \wedge \theta^j &= 0. & (16) \end{aligned}$$

They are zero because we are working in a background where the torsion is zero.

In orthonormal bases,

$$\omega_{\mu\nu} = -\omega_{\nu\mu},$$

as [6] shows in chapter 3, and therefore

$$\begin{aligned} {\omega^0}_i &= {\omega^i}_0 \\ {\omega^i}_j &= \omega_{ij}. \end{aligned}$$

Evaluated on $T(\Sigma)$, (15) and (16) yield

$$\begin{aligned} \left[{\omega^0}_i \wedge \bar{\theta}^i\right]_{T(\Sigma)} &= 0 & (17) \\ \left[d\bar{\theta}^i + {\omega^i}_j \wedge \bar{\theta}^j\right]_{T(\Sigma)} &= 0. & (18) \end{aligned}$$

A lemma from Cartan[7] applied to the first equation (17) leads to the existence of functions $K_{ij} = K_{ji}$ with

$${\omega^0}_i|_{T(\Sigma)} = -K_{ij}\bar{\theta}^j.$$

Equation (18) compared to (16) yields

$${\omega^i}_j|_{T(\Sigma)} = {\bar{\omega}^i}_j.$$

We can calculate the curvature forms with the *second equation of structure*[8]

[5] These equations are derived in [5], p. 68, for example.

[6] A definition of connection one-forms and another derivation of these equations can be found in the comprehensive chapter 3 of [6].

[7] Cartan's lemma can be found in [5], p. 483.

[8] A derivation of the second equation of structure can be found in [6], chapter 3.

$$\Omega^{\mu}{}_{\nu} = d\omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\sigma} \wedge \omega^{\sigma}{}_{\nu}, \tag{19}$$

Note that d is the exterior derivative.

It is shown above that $\Omega^0{}_0 = 0$. We only have to calculate $\Omega^i{}_j$ and $\Omega^i{}_0$.

$$\begin{aligned}
\Omega^i{}_j|_{T\Sigma)} &= \left[d\omega^i{}_j + \omega^i{}_s \wedge \omega^s{}_j + \omega^i{}_0 \wedge \omega^0{}_j\right]|_{T(\Sigma)} \\
&= d\bar{\omega}^i{}_j + \bar{\omega}^i{}_s \wedge \bar{\omega}^s{}_j + \left(-\eta^{i\lambda} K_{\lambda s}\bar{\theta}^s\right) \wedge \left(-K_{jt}\bar{\theta}^t\right) \\
&= \bar{\Omega}^i{}_j + \left(-K^i{}_s\bar{\theta}^s\right) \wedge \left(-K_{jt}\bar{\theta}^t\right) \\
&= \bar{\Omega}^i{}_j + K^i{}_s K_{jt}\bar{\theta}^s \wedge \bar{\theta}^t
\end{aligned}$$

And likewise

$$\Omega^0{}_i|_{T(\Sigma)} = -\bar{D}K_{ij} \wedge \bar{\theta}^j,$$

with $\bar{D}K_{ij} = K_{ij}|_s\bar{\theta}^s := \bar{\nabla}_s K_{ij}\bar{\theta}^s$.

We insert these results into the G_{00} equation (14) and R_{0i} equation (13) and find

$$\begin{aligned}
G_{00} &= \frac{1}{2}\Omega^k{}_i(\bar{e}_k, \bar{e}_i) \\
&= \frac{1}{2}\left[\bar{\Omega}^k{}_i + K^k{}_s K_{it}\bar{\theta}^s \wedge \bar{\theta}^t\right](\bar{e}_k, \bar{e}_i) \\
&= \frac{1}{2}\left[\left(\bar{\Omega}^k{}_i\right)(\bar{e}_k, \bar{e}_i) + \left(K^k{}_s K_{it}\bar{\theta}^s \wedge \bar{\theta}^t\right)(\bar{e}_k, \bar{e}_i)\right] \\
&= \frac{1}{2}\left[\sum_{i=1}^{3}\bar{R}_{ii} + K^k{}_s K_{it}\left(\bar{\theta}^s\bar{e}_k\bar{\theta}^t\bar{e}_i - \bar{\theta}^s\bar{e}_i\bar{\theta}^t\bar{e}_k\right)\right] \\
&= \frac{1}{2}\left[\bar{R} + K^k{}_s K_{it}\left(\delta^s{}_k\delta^t{}_i - \delta^s{}_i\delta^t{}_k\right)\right] \\
&= \frac{1}{2}\left[\bar{R} + K^k{}_k K_{tt} - K^k{}_i K_{ik}\right] \\
&= \frac{1}{2}\left[\bar{R} + K^k{}_k K^t{}_t - K^k{}_i K^i{}_k\right] \\
&= \frac{1}{2}\left[\bar{R} + [tr(K)]^2 - tr(K^2)\right],
\end{aligned} \tag{20}$$

$$\begin{aligned}
R_{0i} &= \Omega^0{}_j(\bar{e}_j, \bar{e}_i) \\
&= -\bar{D}K_{js} \wedge \bar{\theta}^s(\bar{e}_j, \bar{e}_i) \\
&= -\bar{\nabla}_l K_{js}\bar{\theta}^l \wedge \bar{\theta}^s(\bar{e}_j, \bar{e}_i) \\
&= -\bar{\nabla}_l K_{js}\left(\delta^l{}_j\delta^s{}_i - \delta^l{}_i\delta^s{}_j\right) \\
&= -\bar{\nabla}_j K_{ji} + \bar{\nabla}_i K_{jj} \\
&= \bar{\nabla}_i K^j{}_j - \bar{\nabla}_j K^j{}_i.
\end{aligned} \tag{21}$$

As usual, tr denotes the trace of a matrix.

2.2.2 Cartan's Equations Of Structure Solved

To find R_{00} and R_{ij}, we have to solve the first equation of structure (15) and (16). To find the $\omega^0{}_j$s and $\omega^i{}_j$s, we first write the connection one-forms in the 3+1 split:

$$\omega^\mu{}_\nu = \omega^\mu{}_\nu(e_0)\theta^0 + \omega^\mu{}_\nu(e_i)\theta^i.$$

With our bases defined in (4) and the last section, we have

$$\begin{aligned}
\omega^0{}_j &= \omega^0{}_j(e_0)\theta^0 + \underbrace{\omega^0{}_j(\bar{e}_i)}_{-K_{ij}}\theta^i \\
&= -K_{ij}\theta^i + \omega^0{}_j(e_0)\theta^0 && (22) \\
\omega^i{}_j &= \omega^i{}_j(e_0)\theta^0 + \omega^i{}_j(\bar{e}_i)\theta^i \\
&= \omega^i{}_j(e_0)\theta^0 + \omega^i{}_j(\bar{e}_k)\left[\bar{\theta}^k + \frac{1}{\alpha}\beta^k\theta^0\right] \\
&= \bar{\omega}^i{}_j + \left[\frac{1}{\alpha}\bar{\omega}^i{}_j(\beta) + \omega^i{}_j(e_0)\right]\theta^0. && (23)
\end{aligned}$$

Inserting (22) into the first equation of structure (15) yields

$$\omega^0{}_j(e_0) = \frac{1}{\alpha}\bar{\nabla}_j\alpha,$$

and with (22),

$$\omega^0{}_j = -K_{ij}\theta^i + \frac{1}{\alpha}\bar{\nabla}_j\alpha.$$

Remains to find the $\omega^i{}_j$s. To calculate these, we need $d\theta^i$. We define

$$c^i{}_j\bar{\theta}^j := \partial_t\bar{\theta}^i,$$

note that $d = \bar{d} + dt \wedge \partial_t$, and find

$$\begin{aligned}
d\theta^i &= \bar{d}\bar{\theta}^i + dt \wedge \partial_t\bar{\theta}^i + \bar{d}\beta^i \wedge dt + 0 \\
&= -\bar{\omega}^i{}_j \wedge \bar{\theta}^j + \frac{1}{\alpha}\theta^0 \wedge \left[c^i{}_j\bar{\theta}^j - \bar{d}\beta^i\right].
\end{aligned}$$

Inserting these $d\theta^i$s and (23) into the first equation of structure leads to

$$\begin{aligned}
\left[K_{ij} + \omega_{ij}(e_0) + \frac{1}{\alpha}\bar{\omega}_{ij}(\beta)\right]\theta^j \wedge \theta^0 &= -c_{ij}\frac{1}{\alpha}\theta^j \wedge \theta^0 \\
&\quad + \underbrace{\frac{1}{\alpha}\bar{d}\beta^j \wedge \theta^0 + \bar{\omega}^i{}_j \wedge \underbrace{(\theta^j - \bar{\theta}^j)}_{\frac{1}{\alpha}\beta^j}}_{\frac{1}{\alpha}\bar{D}\beta^j\wedge\theta^0}
\end{aligned}$$

$\Longrightarrow$

$$\omega_{ij}(e_0) + K_{ij} = \frac{1}{\alpha}\left[\beta_{i|j} - c_{ij} - \bar{\omega}_{ij}(\bar{\beta})\right],$$

where we used $_{|j}$ as ∇_j. With the useful notation $\omega_{(ij)} := \frac{1}{2}(\omega_{ij} + \omega_{ji})$ and $\omega_{[ij]} := \frac{1}{2}(\omega_{ij} - \omega_{ji})$ we can write the previous equation as

$$\begin{aligned} K_{ij} &= \frac{1}{\alpha}\left[\beta_{(i|j)} - c_{(ij)}\right] && (24) \\ \omega_{ij}(e_0) &= \frac{1}{\alpha}\left[\beta_{[i|j]} - c_{[i|j]} - \bar{\omega}_{ij}(\bar{\beta})\right]. && (25) \end{aligned}$$

Now that we have found the connection one-forms ω_{ij}, we are able to calculate the remaining ${\Omega^0}_j$s and the $d\theta^i$s using the equations of structure and the K_{ij}s. Since only the rearranging of the evolving terms is rather complex and there is nothing new to the calculation we have already made above, we only state the results here:

$$\begin{aligned} R_{00} &= \frac{1}{\alpha}\bar{\Delta}\alpha + \frac{1}{\alpha}\left(\partial_t - \bar{L}_{\bar{\beta}}\right) tr(K) - tr(K^2) \\ R &= \bar{R} + [tr(K)]^2 + tr(K^2) - \frac{2}{\alpha}\bar{\Delta}\alpha - \frac{2}{\alpha}\left(\partial_t - \bar{L}_{\bar{\beta}}\right) tr(K) \\ R_{ij} &= \bar{R}_{ij} + {K^s}_s K_{ij} - 2{K_i}^s K_{sj} - \frac{1}{\alpha}\bar{\nabla}_i\bar{\nabla}_j\alpha - \frac{1}{\alpha}\left([\partial_t - \bar{L}_{\bar{\beta}}]K\right)_{ij}, \end{aligned}$$

where L_β is the Lie-derivative.[9]

[9] The Lie-derivative is defined in [5], p. 23, for example.

3 Perturbation Equations

This chapter contains the derivation of the perturbed Einstein equations for the Robertson-Walker universe. To compose these, we first calculate the left side of the equation, consisting of the perturbed Ricci tensor and the perturbed Ricci scalar. Afterwards, the perturbation of the right side of Einstein's equations will be found rather easily. We pay special attention to gauge invariance, since all terms appearing after a gauge transformation have the potential of being unphysical.[10]

3.1 General Assumptions And Perturbed Metric

In the last section, we have calculated all quantities necessary to compose the Einstein tensor $G_{\mu\nu}$ in the 3+1 formalism. We will use the 3+1 idea also to deal with perturbations.

At the heart of our calculations lies the assumption that our universe is homogeneous and isotropic on large scales and can be described with theories where distances are measured according to the Robertson-Walker metric, which looks like

$$g = -a(t)^2 \left[dt^2 - \gamma_{ijdx^i dx^j}\right] \tag{26}$$

in its most general form. This corresponds to a 3+1 split, where the γ_{ij} metric lives on the Σ spaces defined in the first section, three-dimensional spaces with constant curvature. To factor out $a(t)^2$, we have to write this equation in the conformal time t (usally denoted η, but we will stick to t because it makes the following equations a little more familiar). Remember that we can write the line element

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + a(t)^2 d\sigma^2 \\ &= -a(t)^2(\eta^2 - d\sigma^2), \end{aligned}$$

where $\eta := \frac{dt}{a}$ is the conformal time. Appendix A contains the derivation of Einstein's equations in this metric.

Of course, there are other metrics that describe our universe well on big scales. But the crucial advantage of using the Robertson-Walker metric is the fact that we are then calculating on a sphere and we can look up in a handbook of mathematics that for every sphere, there exists a basis $Y_{(k)}$ of the harmonic functions of the sphere. This means that

$$(\Delta^{(\gamma)} + k^2)Y_{(k)} = 0. \tag{27}$$

One uses the terms *scalar perturbations*, *vector perturbations* and *tensor perturbations* to distinguish perturbations according to their behavior under

[10] See [1], p. 3 and p. 40 for more on problems associated with gauge transformations and a complete account of useful gauges in cosmological perturbation theory.

coordinate transformations. In this paper, we are only looking at *density fluctuations*, this means we will only have to look at scalar perturbations, since the perturbation equations decouple.

What will perturbations of scalar, vector and tensor quantities look like, given that $Y_{(k)}$ is a basis of the harmonic functions of the sphere we are working on? There is an immediate answer to this question: Perturbations of scalar quantities can be developed in terms of $Y_{(k)}$, perturbations of vectors in $\nabla_i^{(\gamma)} Y_{(k)}$ and perturbations of tensors over $T(\Sigma)$ in $Y \cdot \gamma_{ij}$ or $\nabla_i^{(\gamma)} \nabla_j^{(\gamma)} Y_{(k)}$.[11]

Therefore, scalar perturbations of the Roberton-Walker metric are parameterized by four functions, which are historically called ϕ, ψ, B, and E.

Throughout the paper, δ without indices denotes perturbed quantities, e.g. the metric is written as

$$g = \underbrace{g^{(0)}}_{\text{0th order}} + \underbrace{\delta g}_{\text{1st order}} . \tag{28}$$

The coefficients of (26) are

$$\begin{aligned} g_{tt}^{(0)} &= -a^2 & (29) \\ g_{ti}^{(0)} &= 0 & (30) \\ g_{ij}^{(0)} &= a^2 \gamma_{ij} . & (31) \end{aligned}$$

With the parameterization by ϕ, ψ, B, and E, we make the following Ansatz:

$$\begin{aligned} \delta g_{tt} &= -2a^2 \phi & (32) \\ \delta g_{ti} &= -a^2 \nabla_i B & (33) \\ \delta g_{ij} &= -2a^2 \left[\psi \gamma_{ij} - \nabla_i \nabla_j E \right] . & (34) \end{aligned}$$

Then, the most general scalar perturbation of the metric is

$$\delta g = a^2 \left[-2\phi dt^2 - 2\nabla_i B dx^i dx^t - (2\psi \gamma_{ij} - \nabla_i \nabla_j E) \, dx^i dx^j \right] . \tag{35}$$

We wrote ∇_i instead of $\nabla_i^{(0)}$, and Δ instead of $\Delta^{(\gamma)}$ - we remember that we are working on the Σs in what follows.

Now, (10) is $g = g^{(0)} + \delta g$ in the 3+1 formulation. From there, it follows, that

$$\begin{aligned} g_{tt} &= -\alpha^2 & (36) \\ g_{ti} &= \beta^i . & (37) \end{aligned}$$

[11] A mathematical proof of this can be found in [1], p. 139.

Because (and that is an idea we will use frequently), according to (28), we must treat δ like a derivation, meaning

$$\begin{aligned} g &= g^{(0)} + \delta g \\ \Longrightarrow g_{tt} &= g^{(0)}_{tt} + \delta g_{tt} \\ -\alpha^2 &= -(\alpha^{(0)} + \delta\alpha)^2 \\ &= (\alpha^{(0)})^2 - 2\alpha^{(0)}\delta\alpha + (\delta\alpha)^2, \end{aligned}$$

we get through comparison (leaving out the $(\delta\alpha)^2$ to first order)

$$\delta g_{tt} = -2\alpha\delta\alpha,$$

so we must have $\frac{\delta\alpha}{\alpha^{(0)}} = \phi$, because of our Ansatz (32).

The perturbations of the inverse metric are easy to calculate with

$$\delta g^{\mu\nu} = -g^{(0)\mu\sigma} g^{(0)\nu\rho} \delta g_{\sigma\rho}, \tag{38}$$

which yields to first order, after insertion of the inverse values of (29) and following equations:

$$\begin{aligned} \delta g^{tt} &= \frac{2}{a^2}\phi \\ \delta g^{ti} &= -\frac{1}{a^2}\gamma^{ij}\nabla_i B \\ \delta g^{ij} &= \frac{2}{a^2}\left[\psi\gamma^{ij} - E^{|ij}\right]. \end{aligned} \tag{39}$$

We denoted $\nabla^i\nabla^j$ by $^{|ij}$.

3.2 Gauge Transformations Of $(0,2)$ Tensors

In this context, a gauge transformation equals a coordinate transformation of the perturbed first-order quantities.[12] To first order, it suffices to consider *infinitesimal coordinate transformations*. The simplest infinitesimal coordinate transformation is to add a vector to each point in Σ, which can be realized by adding a vectorfield X^μ,

$$x^\mu \longrightarrow x^\mu + X^\mu,$$

and we imagine this added vector field to perturb the considered system, so we write

$$x^\mu \longrightarrow x^\mu + \delta x^\mu. \tag{40}$$

How does the perturbation of a tensor generally transform under an infinitesimal coordinate transformation caused by a vector field? Consider a $(0,2)$ tensor

[12] [6] provides good background on p. 438.

$$T = T_{\mu\nu} dx^\mu dx^\nu,$$

the perturbation of which looks like (remember that we have to treat δ like a derivation)

$$\begin{aligned}
\delta T &= \delta(T_{\mu\nu} dx^\mu dx^\nu) \\
&= T_{\mu\nu,\sigma} \delta x^\sigma dx^\mu dx^\nu + T_{\sigma\nu} d(\delta x^\sigma) \otimes dx^\nu + T_{\mu\sigma} dx^\mu \otimes d(\delta x^\sigma) \\
&= \left(T_{\mu\nu,\sigma} X^\sigma + T_{\sigma\nu} X^\sigma_{,\mu} + T_{\mu\sigma} X^\sigma_{,\nu}\right) dx^\mu dx^\nu \\
&= L_X T \cdot dx^\mu dx^\nu,
\end{aligned}$$

where we used $d(\delta x^\sigma) = X^\sigma_{,\mu} dx^\mu$. By definition, $L_X T$ is the *Lie derivative of T regarding X*[13]. So the perturbation of a $(0,2)$ tensor T transforms like

$$\delta T \longrightarrow \delta T + L_X T \tag{41}$$

under a gauge transformation. Applied to our metric g, which is also a $(0,2)$ tensor, we get

$$(\delta g)_{\mu\nu} \longrightarrow (\delta g_{\mu\nu}) + X^\sigma g_{\mu\nu,\sigma} + X^\sigma_{,\mu} g_{\sigma\nu} + X^\sigma_{,\nu} g_{\mu\sigma}. \tag{42}$$

For our metric (26), the components of the Lie derivative are

$$(L_X g)_{ij} = X^t g_{ij,t} + \underbrace{X^k g_{ij,k}}_{a^2 \nabla_k \gamma_{ij} = 0} + \underbrace{X^s{}_{,i} g_{sj} + X^s{}_{,j} g_{is}}_{a^2[\gamma_{is}\nabla_j X^s + \gamma_{js}\nabla_i X^s]}, \tag{43}$$

which yields for the 3+1 formulation

$$\begin{aligned}
(L_X g)_{tt} &= X^t g_{tt,t} + a^2 \left[\gamma_{ts} \nabla_t X^s + \gamma_{ts} \nabla_t X^s\right] \\
&= X^t g_{tt,t} + 2a^2 \gamma_{tt} \partial_t X^t \\
&= -X^t \cdot 2a\dot{a} - 2a^2 \dot{X}^t \\
&= -2a^2 \left[\dot{X}^t - H X^t\right] \qquad (44) \\
(L_X g)_{it} &= -a^2 \left[X^t{}_{,i} - \gamma_{ij} \dot{X}^j\right] \qquad (45) \\
(L_X g)_{ij} &= 2a^2 \left[H \gamma_{ij} X^t + \gamma_{s(i} X^s_{,j)}\right], \qquad (46)
\end{aligned}$$

where $\frac{\dot{a}}{a} := H$, as usual in cosmology.

It is not hard to imagine that for scalar perturbations there must exist functions T and L such that

$$X^t := T, \qquad X^i := \gamma^{ij} L_{,j}.$$

In this case we write the Lie derivatives of the metric like

[13] Refer [5], p. 23, for a general definition.

$$
\begin{aligned}
(L_X g)_{tt} &= -2a^2\left[\dot{T} + HT\right] = -2a\,[aT]^{\cdot} \\
(L_X g)_{it} &= -a^2\left[T - \dot{L}\right]_{,i} \\
(L_X g)_{ij} &= 2a^2\left[\gamma_{ij} HT + \nabla_i \nabla_j L\right].
\end{aligned}
$$

We can determine how our potentials ϕ, ψ, E, and B from the previous section transform under gauge transformations:

$$
\begin{aligned}
\delta g_{tt} &\longrightarrow \delta g_{tt} + (L_X g)_{tt} \\
\Longrightarrow -2a^2\phi &\longrightarrow -2a^2\left[\phi + \dot{T} + HT\right] \\
\delta g_{ti} &\longrightarrow \delta g_{ti} + (L_X g)_{ti} \\
\Longrightarrow -a^2 B_{|i} &\longrightarrow -a^2\left[B + T - \dot{L}\right]_{|i} \\
\delta g_{ij} &\longrightarrow \delta g_{ij} + (L_X g)_{ij} \\
\Longrightarrow -2a^2\left[\psi\gamma_{ij} - \nabla_i\nabla_j E\right] &\longrightarrow -2a^2\left[(\psi - HT)\gamma_{ij} - \nabla_i\nabla_j(E + L)\right].
\end{aligned}
$$

Therefore, the four potentials transform like

$$
\begin{aligned}
\phi &\longrightarrow \phi + \left[\dot{T} + HT\right] && (47)\\
\psi &\longrightarrow \psi - HT && (48)\\
B &\longrightarrow B + \left[T - \dot{L}\right] && (49)\\
E &\longrightarrow E + L. && (50)
\end{aligned}
$$

3.3 Gauge Invariant Bardeen Potentials

With the four potentials ϕ, ψ, B, and E and the two *gauge freedoms* T and L, one can find two *gauge invariant potentials*, which of course are not unique. The simplest two gauge invariant potentials are the *Bardeen potentials* Φ and Ψ:

$$
\begin{aligned}
\Phi &:= \phi - \frac{1}{a}\left[a(\dot{E} + B)\right]^{\cdot} && (51)\\
\Psi &:= \psi + \frac{\dot{a}}{a}\left[\dot{E} + B\right]. && (52)
\end{aligned}
$$

These gauge invariant potentials are crucial to cosmological perturbation theory. We will use them to find gauge invariant quantities of matter.

Two short calculations show that the Bardeen potentials are really gauge invariant. Using equations (47) to (50) in the Bardeen potentials, we have

$$
\phi - \frac{1}{a}\left[a(\dot{E} + B)\right]^{\cdot} \longrightarrow \phi + \dot{T} + \frac{\dot{a}}{a}T - \frac{1}{a}\left[a\ \left(\dot{E} + \dot{L} + B + T - \dot{L}\right)\right]^{\cdot}
$$

$$
\begin{aligned}
&= \phi + \dot{T} + \frac{\dot{a}}{a}T - \frac{1}{a}\left[\dot{a}\left(\dot{E} + \dot{L} + B + T - \dot{L}\right)\right. \\
&\quad \left. + a\left(\ddot{E} + \ddot{L} + \dot{B} + \dot{T} - \ddot{L}\right)\right] \\
&= \phi + \dot{T} + \frac{\dot{a}}{a}T - \frac{\dot{a}}{a}\left(\dot{E} + B + T\right) \\
&\quad - (\ddot{E} + \dot{B}) - \ddot{T} \\
&= \phi - \frac{1}{a}\left[a(\dot{E} + B)\right]^{\cdot} \\
\Longrightarrow \Phi &\longrightarrow \Phi
\end{aligned}
$$

$$
\begin{aligned}
\psi + \frac{\dot{a}}{a}\left[\dot{E} + B\right] &\longrightarrow \psi - H\dot{T} + \frac{\dot{a}}{a}\left[\dot{E} + \dot{L} + B + T - \dot{L}\right] \\
&= \psi - \frac{\dot{a}}{a}T + \frac{\dot{a}}{a}\left[\dot{E} + B\right] + \frac{\dot{a}}{a}T \\
\Longrightarrow \Psi &\longrightarrow \Psi.
\end{aligned}
$$

3.3.1 The Longitudinal Gauge $E = B = 0$

If we use our gauge freedoms T and L and define them like

$$L := -E, \qquad T := -\dot{E} - B, \tag{53}$$

we see from (49) and (50) that

$$E \longrightarrow 0, \qquad B \longrightarrow 0.$$

Obviously, in this case we can set $E = 0$ and $B = 0$, because they will be zero after the gauge transformation anyway - but then we have chosen a gauge: We have lost our gauge freedoms L and T, defining them as above. This is called the *longitudinal gauge*. It will make our calculations much easier that we can drop some terms in the δgs, when we are calculating the perturbations of $\delta G^{\mu}{}_{\nu}$ in the longitudinal gauge (see (32) to (34)). And the result will immediately be gauge invariant, because in this gauge

$$\Phi = \phi_{\text{longitudinal}}, \qquad \Psi = \psi_{\text{longitudinal}}. \tag{54}$$

Of course, there are many other gauges fit to other needs.[14]

3.4 Perturbations Of The Einstein Tensor

This section will provide the $\delta G^{\mu}{}_{\nu}$ equations, the left side of the perturbed Einstein equation. Analogous to (28), we write

$$G = G^{(0)} + \delta G.$$

We are now choosing the longitudinal gauge defined in the previous section. In this gauge, the perturbations of the metric are

[14] See [1], p. 40 and following pages for an overview of useful gauges and their applications.

$$\begin{aligned}
\delta g_{tt} &= -2a^2\phi \\
\delta g_{ti} &= 0 \\
\delta g_{ij} &= 2a^2\left[\psi\gamma_{ij}\right] = -g_{ij}^{(0)}2\psi,
\end{aligned}$$

which we get from the δgs within our Ansatz (32) to (34) with $E = B = 0$. We do not have to worry about gauges, since our results will by themselves be gauge invariant according to (54).

From the 3+1 split, we know that a metric g has the general form (10):

$$g = \left[-\alpha^2 + \beta^i\beta_i\right]dt^2 + 2\beta_i dt dx^i + \bar{g}_{ij}dx^i dx^j.$$

Working with the Robertson-Walker metric (26), we know there are no off-diagonal terms g_{ti}, so the βs must be zero (which is in accordance with our notion that time stands normal to the Σs: the shift vector on Σ vanishes). So we are actually working with a metric of the form

$$g = -\alpha^2 dt^2 + g_{ij}dx^i dx^j. \tag{55}$$

From now on we leave out the bar that distinguished quantities on Σ - otherwise, we would have had $\bar{g}_{ij}$ in the previous equation.

Looking at (29), we have

$$-(\alpha^{(0)})^2 = g_{tt}^{(0)} = -a^2 \Longrightarrow \alpha^{(0)} = a,$$

and with $\phi = \frac{\delta\alpha}{\alpha^{(0)}}$,

$$\delta\alpha = a\phi. \tag{56}$$

We raise the first index in the equations for Einstein tensor, Ricci tensor and Ricci scalar, found in the 3+1 formalism, and have the equations we start with:

$$\begin{aligned}
G^t{}_t &= -\frac{1}{2}R(g) + \frac{1}{2}\left[K^i{}_j K^j{}_i - K^i{}_i K^j{}_j\right] && (57)\\
G^t{}_i &= \frac{1}{\alpha}\left(\nabla_j K^j{}_i - \nabla_i K^j{}_j\right) && (58)\\
R^i{}_j &= R^i{}_j(g) + K^s{}_s K^i{}_j - 2K^i{}_s K^s{}_j - \frac{1}{\alpha}\left[\nabla^i\nabla_j\alpha + g^{is}\dot{K}_{js}\right] && (59)\\
R &= R(g) + K^i{}_j K^j{}_i + K^i{}_i K^j{}_j - \frac{2}{\alpha}\left[\Delta\alpha + \dot{K}^s{}_s\right]. && (60)
\end{aligned}$$

Our task is now to calculate the perturbations of every single term, which means we have to calculate the perturbations of the K_{ij}s and the R_{ij}s on Σ. Then, we can compose the perturbations of (57) to (60) and we are finished with the perturbations of the left side of Einstein's equations.

3.4.1 Perturbation Of K_{ij}

We only have given and we only need the Robertson-Walker metric to calculate the Einstein tensor, and since G is composed of the K_{ij}s, we can express the K_{ij}s in terms of the metric.

We make use of the so-called *0th equation of structure*[15] (remember Cartan's equations of structure from the preliminaries, this is another one that is quite similar):

$$dg_{ik} = \omega_{ik} + \omega_{ki}. \tag{61}$$

On the left side, we have the total differential of g, and since the metric is only dependent on time,

$$dg_{ik} = \dot{g}_{ik} dt = \dot{g}_{ik} \frac{\theta^0}{\alpha},$$

with (4). Now we calculate

$$\begin{aligned}
\dot{g}_{ik} \frac{\theta^0}{\alpha} &= \omega_{ik} + \omega_{ki} \\
\Longrightarrow \frac{1}{\alpha} \dot{g}_{ik} \underbrace{\theta^0(e_0)}_{1} &= \omega_{ik}(e_0) + \omega_{ki}(e_0) \\
&= -K_{ik} + \frac{1}{\alpha}\left[\beta_{i|k} - c_{ik} - \bar{\omega}_{ik}(\bar{\beta})\right] \\
&\quad -K_{ki} + \frac{1}{\alpha}\left[\beta_{k|i} - c_{ki} - \bar{\omega}_{ki}(\bar{\beta})\right] \\
&= -2K_{ik} \\
\Longrightarrow K_{ij} &= -\frac{1}{2\alpha} \dot{g}_{ij}.
\end{aligned} \tag{62}$$

Here, we used a lot of facts we ran across before: (24) and (25), $d(d\bar{\theta}^i) = d(dx^i) = 0 \Longrightarrow c_{ik} = 0$, and we remembered that we are working in the longitudinal gauge, where there must not exist any βs in metric g (55), so the $\bar{\omega}(\bar{\beta})$s and $\beta_{i|k}$s are zero.

Since $g_{ij}^{(0)} = a^2 \gamma_{ij}$, we have

$$\dot{g}_{ij}^{(0)} = 2a\dot{a}\gamma_{ij} = 2Ha^2\gamma_{ij} = 2Hg_{ij}^{(0)}.$$

It follows to 0th order

$$\begin{aligned}
K_{ij}^{(0)} &= -\frac{1}{2\alpha^{(0)}} \dot{g}_{ij}^{(0)} = -\frac{1}{2a} 2a\dot{a}\gamma_{ij} = -\dot{a}\gamma_{ij} \\
\Longrightarrow K_{ij}^{(0)} &= -\frac{1}{a} H g_{ij}^{(0)}.
\end{aligned} \tag{63}$$

In the longitudinal gauge,

[15] This equation is derived in [5], p. 62 and following pages.

$$\begin{aligned}
\delta\dot{g}_{ij} &= \left[-2a^2\psi\gamma_{ij}\right]^{\cdot} \\
&= -2\cdot 2a\dot{a}\psi\gamma_{ij} - 2a^2\dot{\psi}\gamma_{ij} \\
&= -2\psi\dot{g}_{ij}^{(0)} - 2\dot{\psi}g_{ij}^{(0)} \\
&= -2\psi 2Hg_{ij}^{(0)} - 2\dot{\psi}g_{ij}^{(0)} \\
&= -2g_{ij}^{(0)}\left[\dot{\psi} + 2H\psi\right],
\end{aligned}$$

and therefore, because the perturbation of K_{ij} is the first order term of (62) expanded around $K_{ij}^{(0)}$:

$$\begin{aligned}
\delta K_{ij} &= -\underbrace{\delta\left(\frac{1}{2\alpha}\right)}_{-\frac{\delta\alpha}{2(\alpha^{(0)})^2}}\ \underbrace{\dot{g}_{ij}^{(0)}}_{2Hg_{ij}^{(0)}} - \frac{1}{2\alpha}\delta\dot{g}_{ij} \\
&= \frac{1}{\alpha^{(0)}}\left[\underbrace{\frac{\delta\alpha}{\alpha^{(0)}}}_{\phi}2Hg_{ij}^{(0)} + 2g_{ij}^{(0)}\left(\dot{\psi}+2H\psi\right)\right] \\
&= \frac{1}{a}g_{ij}^{(0)}\left[H\phi + \dot{\psi} + 2H\psi\right].
\end{aligned}$$

We use this result and $g = g^{(0)} + \delta g$ to calculate

$$\begin{aligned}
\delta K^i{}_j &= \delta\left(g^{il}K_{lj}\right) \\
&= \delta g^{il}K_{lj} + g^{il}\delta K_{lj} \\
&= \delta g^{il}\left[K_{lj}^{(0)} + \delta K_{lj}\right] + \left[g^{(0)il} + \delta g^{il}\right]\delta K_{lj} \\
&= -\frac{2}{a^2}\psi\gamma^{il}\frac{1}{a}Hg_{lj}^{(0)} + \frac{1}{a^2}\gamma^{il}\frac{1}{a}g_{lj}^{(0)}\left[H\phi + \dot{\psi} + 2H\psi\right] \\
&= \frac{1}{a^3}\gamma^{il}g_{lj}^{(0)}\left[H\phi + \dot{\psi}\right] \\
&= \frac{1}{a^3}a^2\gamma^{il}\gamma_{lj}\left[H\phi + \dot{\psi}\right] \\
&= \frac{1}{a}\delta^i{}_j\left[H\phi + \dot{\psi}\right].
\end{aligned}$$

Notice that we can leave out all $\delta \ldots \cdot \delta \ldots$ terms since they would not yield something not to first order anymore. The calculation for δK^{ij} is almost the same. Consolidated, we find for the perturbations of K_{ij}:

$$\delta K_{ij} = \frac{1}{a}g_{ij}^{(0)}\left[H\phi + \dot{\psi} + 2H\psi\right] \tag{64}$$

$$\delta K^i{}_j = \frac{1}{a}\delta^i{}_j\left[H\phi + \dot{\psi}\right] \tag{65}$$

$$\delta K^{ij} = \frac{1}{a}g^{(0)ij}\left[H\phi + \dot{\psi} - 2H\psi\right]. \tag{66}$$

Looking at (57) to (60), we also need the perturbations of the $K^s{}_s K^i{}_j$s and the $K^i{}_s K^s{}_i$s:

$$\begin{aligned}
\delta(K^s{}_s K^i{}_j) &= \delta K^s{}_s K^i{}_j + K^s{}_s \delta K^i{}_j \\
&= \delta K^s{}_s \left[K_j^{(0)i} + \delta K^i{}_j\right] + \left[K_s^{(0)s} + \delta K^s{}_s\right] \delta K^i{}_j \\
&= \delta K^s{}_s g^{il} K_{lj}^{(0)} + g^{sk} K_{ks}^{(0)} \delta K^i{}_j \\
&= \delta K^s{}_s \left[g^{(0)il} + \delta g^{il}\right] K_{lj}^{(0)} + \left[g^{(0)sk} + \delta g^{sk}\right] K_{ks}^{(0)} \delta K^i{}_j \\
&= -\frac{1}{a}\delta^s{}_s(H\phi + \dot{\psi}\frac{1}{a^2}\gamma^{il}\frac{1}{a}H \underbrace{g_{lj}^{(0)}}_{a^2\gamma_{lj}} \\
&\quad -\frac{1}{a^2}\gamma^{sk}\frac{1}{a}H \underbrace{g_{ks}^{(0)}}_{a^2\gamma_{ks}} \frac{1}{a}\delta^i{}_j \left(H\phi + \dot{\psi}\right) \\
&= -\frac{1}{a}H\left(H\phi + \dot{\psi}\right)\delta^i{}_j \underbrace{[\delta^s{}_s + \delta^s{}_s]}_{3+3} \\
&= -\frac{6H}{a^2}\left(H\phi + \dot{\psi}\right)\delta^i{}_j .
\end{aligned}$$

A similar calculation for $K^i{}_s K^s{}_i$ gives almost the same result. We have

$$\delta(K^s{}_s K^i{}_j) = -\frac{6H}{a^2}\left(H\phi + \dot{\psi}\right)\delta^i{}_j \tag{67}$$

$$\delta(K^i{}_s K^s{}_i) = -\frac{6H}{a^2}\left(H\phi + \dot{\psi}\right). \tag{68}$$

The necessary derivations of the 0th order terms are seen to be

$$\left[K_{ij}^{(0)}\right]^{\cdot} = -\frac{1}{a}\left[\dot{H} + H^2\right] g_{ij}^{(0)} \tag{69}$$

$$\begin{aligned}
\left[K_i^{(0)i}\right]^{\cdot} &= \left[(g^{(0)il} + \delta g^{il})K_{li}^{(0)}\right]^{\cdot} \\
&= \left[-\frac{1}{a^2}\gamma^{il}\frac{1}{a}\frac{\dot{a}}{a}a^2\gamma_{li}\right]^{\cdot} \\
&= \left[-\frac{\dot{a}}{a^2}\underbrace{\delta^i{}_i}_{3}\right]^{\cdot} = -3\left(\frac{\ddot{a}}{a^2} - 2\frac{\dot{a}^2}{a^3}\right) \\
&= \frac{3}{a}\left[H^2 - \dot{H}\right],
\end{aligned} \tag{70}$$

and the 1st order terms become in the same manner

$$\delta\dot{K}^i{}_i = \frac{3}{a}\left[(\dot{H} - H^2)\phi + H(\dot{\phi} - \dot{\psi}) + \ddot{\psi}\right] \tag{71}$$

$$\delta\dot{K}^i{}_j = \delta\left(g^{is}\dot{K}_{sj}\right) = \frac{1}{a}\left[(\dot{H} + H^2)\phi + H(\dot{\phi} + 3\dot{\psi}) + \ddot{\psi}\right]\delta^i{}_j . \tag{72}$$

From (69) and (70), we get directly,

$$\left(\frac{1}{\alpha}\dot{K}^i{}_i\right)^{(0)} = \frac{3}{a^2}(H^2 - \dot{H}) \tag{73}$$

$$\left(\frac{1}{\alpha}g^{is}\dot{K}_{sj}\right)^{(0)} = -\frac{1}{a^2}(\dot{H} + H^2)\delta^i{}_j. \tag{74}$$

The perturbation of the first equation is, if we consider the last few equations,

$$\begin{aligned}
\delta\left(\frac{1}{\alpha}\dot{K}^i{}_i\right)^{(0)} &= \delta\left(\frac{1}{\alpha}\dot{K}^i{}_i\right)^{(0)} + \left(\frac{1}{a}\delta\dot{K}^i{}_i\right)^{(0)} \\
&= -\frac{1}{\alpha^{(0)}}\frac{\delta\alpha}{\alpha}\dot{K}_i^{(0)i} + \left(\frac{1}{a}\delta\dot{K}^i{}_i\right)^{(0)} \\
&= -\frac{1}{a}\phi\frac{3}{a}\left[H^2 - \dot{H}\right] \\
&\quad + \frac{1}{a}\frac{3}{a}\left[(\dot{H} - H^2)\phi + H(\dot{\phi} - \dot{\psi}) + \ddot{\psi}\right] \\
&= \frac{3}{a^2}\left[\dot{H} - H^2\right]\phi + \frac{3}{a^2}\left[(\dot{H} - H^2)\phi + H(\dot{\phi} - \dot{\psi}) + \ddot{\psi}\right] \\
\Longrightarrow \delta\left(\frac{1}{\alpha}\dot{K}^i{}_i\right)^{(0)} &= \frac{3}{a^2}\left[2(\dot{H} - H^2)\phi + H(\dot{\phi} - \dot{\psi}) + \ddot{\psi}\right],
\end{aligned}$$

and the perturbation of the second equation is analogous. This yields

$$\delta\left(\frac{1}{\alpha}\dot{K}^i{}_i\right)^{(0)} = \frac{3}{a^2}\left[2(\dot{H} - H^2)\phi + H(\dot{\phi} - \dot{\psi}) + \ddot{\psi}\right] \tag{75}$$

$$\delta\left(\frac{1}{\alpha}g^{is}\dot{K}_{sj}\right)^{(0)} = \frac{1}{a^2}\left[2(\dot{H} + H^2)\phi + H(\dot{\phi} + \dot{\psi}) + \ddot{\psi}\right]. \tag{76}$$

Finally, we need the covariant derivative of K_{ij} in 0th and 1st order. But this is not so complicated as it may seem, because from (62), we get

$$K_{ij}^{(0)} = -\frac{1}{a}\frac{\dot{a}}{a^2}a^2\gamma_{ij} \Longrightarrow \nabla_k K_{ij}^{(0)} = 0,$$

and therefore, to first order,

$$\begin{aligned}
\delta\left(\nabla_j K^j{}_i\right) &= \nabla_j \delta K^j{}_i \\
&= \nabla_j \frac{1}{a}\left(H\phi + \dot{\psi}\right)\delta^j{}_i = \nabla_i \frac{1}{a}\left(H\phi + \dot{\psi}\right).
\end{aligned} \tag{77}$$

These are all necessary perturbations in K_{ij} to compose the perturbation of the Einstein tensor, once we know the perturbations of the Ricci tensor - and this will be our next aim.

3.4.2 Perturbation Of R_{ij}

We borrow the following equation:[16]

$$R_{ij}^{(0)}(g_{ij}^{(0)}) = 2K\gamma_{ij}. \tag{78}$$

With the following idea, we can calculate the 1st order quantities rather elegantly: A *conformal transformation* is a transformation where

$$g_{ij}^{(0)} \longrightarrow \Omega^2 g_{ij}^{(0)} =: \widetilde{g}_{ij}. \tag{79}$$

Conformal transformations[17] conceive the casual structure, and it is generally true that

$$\begin{aligned} R_{ij}^{(0)} \longrightarrow \widetilde{R}_{ij} &= R_{ij}^{(0)} - \nabla_i^{(0)}\nabla_j^{(0)}(ln\Omega) - g_{ij}^{(0)}\Delta^{(0)}(ln\Omega) \\ &\quad +(\nabla_i^{(0)}ln\Omega)(\nabla_j^{(0)}ln\Omega) - g_{ij}^{(0)}(\nabla^{(0)}ln\Omega)^2. \end{aligned} \tag{80}$$

With $\Omega := 1-\psi$, we consider the conformal transformation

$$\widetilde{g}_{ij} = (1-\psi)^2 g_{ij}^{(0)}, \tag{81}$$

and expand $\widetilde{g}_{ij}$ around $g_{ij}^{(0)}$:

$$\widetilde{g}_{ij} = g_{ij}^{(0)} - 2\psi g_{ij}^{(0)} + \ldots, \tag{82}$$

but with (34) in the longitudinal gauge,

$$g_{ij} = g_{ij}^{(0)} + \delta g_{ij} = g_{ij}^{(0)} - 2\psi g_{ij}^{(0)} + \ldots, \tag{83}$$

So g_{ij} and $\widetilde{g}_{ij}$ are identical to 1st order. It follows immediately that

$$\widetilde{R}_{ij} = R_{ij}^{(0)} + \delta R_{ij} + \ldots = R_{ij} \tag{84}$$

to 1st order. So we find δR_{ij} as the 1st order term of the expansion of $\widetilde{R}_{ij}$ around $R_{ij}^{(0)}$. With

$$ln\Omega = ln(1-\psi) = -\psi + \mathbf{O}(\psi^2),$$

we find according to (80):

$$\begin{aligned} \widetilde{R}_{ij} &= R_{ij}^{(0)} + (\nabla_i^{(0)}\nabla_j^{(0)} + g^{(0)}\Delta)\psi + \mathbf{O}(\psi^2) \\ \Longrightarrow \delta R_{ij} &= (\nabla_i\nabla_j + \gamma_{ij}\Delta)\psi. \end{aligned} \tag{85}$$

[16] From [2], p. 721. It can be applied here because we restricted ourselves to the homogeneous, isotropic universe, and the Latin indices indicate we are working only in three dimensions.

[17] The necessary background about conformal transformations and the idea we use here are contained in [6], appendix D.

This is everything we need to calculate the perturbation of the Ricci scalar $R(g)$ in the right side of (60):

$$\begin{aligned}
\delta R &= \delta(R^i{}_i) = \delta(g^{ij}R_{ij}) \\
&= \delta g^{ij}(R^{(0)ij} + \delta R_{ij}) + (g^{(0)ij} + \delta g^{ij})\delta R_{ij} \\
&= \frac{2}{a^2}\psi\gamma^{ij}2K\gamma_{ij} + \frac{1}{a^2}\gamma^{ij}\left(\nabla_i\nabla_j + \gamma_{ij}\Delta\right)\psi \\
&= \frac{4}{a^2}\psi\underbrace{\delta^i{}_i}_{3}K + \frac{1}{a^2}\left(\underbrace{\gamma^{ij}\nabla_i\nabla_j}_{\Delta} + \underbrace{\delta^i{}_i}_{3}\Delta\right)\psi \\
&= \frac{4}{a^2}\psi\left(3K + \Delta\right). \qquad (86)
\end{aligned}$$

Do not confuse $R(g)$ with the left side of (60), we only left the bars out.

With the same kind of calculation, the perturbation of $\delta R^i{}_j$ is

$$\delta R^i{}_j = \frac{1}{a^2}\psi\left[(4K+\Delta)\delta^i{}_j + \nabla^i\nabla_j\right]. \qquad (87)$$

Now we have got all necessary quantities to compose (57) to (60). Keep in mind that we are working in metric (55), and that we must not consider products of perturbed quantities, since two 1st order terms multiplied yield something in 2nd order.

$$G^t{}_t = -\frac{1}{2}R(g) + \frac{1}{2}\left[K^i{}_jK^j{}_i - K^i{}_iK^j{}_j\right] \qquad (57)$$

$$\begin{aligned}
G^{(0)}_{tt} &= -\frac{1}{2}g^{(0)lm}R^{(0)}_{lm} + \frac{1}{2}\left[g^{(0)il}K^{(0)}_{lj}g^{(0)jl}K^{(0)}_{li} - \left(g^{(0)il}K^{(0)}_{li}\right)^2\right] \qquad (88) \\
&= -\frac{1}{2}\frac{1}{a^2}\gamma^{lm}2K\gamma_{lm} \\
&\quad +\frac{1}{2}\left[\frac{1}{a^2}\gamma^{il}\frac{1}{a}\frac{\dot a}{a}a^2\gamma_{lj}\frac{1}{a^2}\gamma^{jl}\frac{1}{a}\frac{\dot a}{a}a^2\gamma_{lj} - \left(-\frac{1}{a^2}\gamma^{il}\frac{1}{a}\frac{\dot a}{a}a^2\gamma_{li}\right)^2\right] \\
&= -\frac{1}{a^2}\delta^l{}_lK + \frac{1}{2}\frac{1}{a^2} + \frac{1}{2}\frac{1}{a^2}\left(\underbrace{\frac{\dot a^2}{a^2}}_{H^2}\underbrace{\delta^i{}_j\delta^j{}_i}_{\delta^i{}_i} - \underbrace{\frac{1}{a^2}\dot a^2}_{H^2}(\delta^i{}_i)^2\right) \\
&= -\frac{3}{a^2}(K + H^2) \qquad (89)
\end{aligned}$$

$$\begin{aligned}
\delta G^t{}_t &= \delta\left(-\frac{1}{2}R(g) + \frac{1}{2}\left[K^i{}_jK^j{}_i - K^i{}_iK^j{}_j\right]\right) \\
&= -\frac{1}{2}\delta R + \frac{1}{2}\delta\left(K^i{}_jK^j{}_i\right) - \frac{1}{2}\delta\left(K^i{}_iK^j{}_j\right) \\
&= -\frac{1}{2}\frac{4}{a^2}(3K+\Delta)\psi + \frac{1}{2}\left[-\frac{6H}{a^2}\left(H\phi + \dot\psi\right)\right]
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}\left[-\frac{6H}{a^2}\left(H\phi+\dot{\psi}\right)\delta^j{}_j\right] \\
&= -\frac{2}{a^2}(3K+\Delta)\psi-\frac{3H}{a^2}(H\phi+\dot{\psi})+\frac{9H}{a^2}(H\phi+\dot{\psi}) \\
&= \frac{2}{a^2}\left[3H(H\phi+\dot{\psi})-(3K+\Delta)\psi\right] \qquad (90)
\end{aligned}
$$

$$
G^t{}_i = \frac{1}{\alpha}\left(\nabla_j K^j{}_i-\nabla_i K^j{}_j\right) \quad (58)
$$

$$
G_i^{(0)t} = \frac{1}{a}\underbrace{\left(\nabla_j g^{(0)jl}K_{li}^{(0)}-\nabla_i g^{(0)lj}K_{lj}^{(0)}\right)}_{0,\ \text{because } \nabla_i(\text{functions of } a,\gamma_{ij})=0} = 0 \qquad (91)
$$

$$
\begin{aligned}
\delta\left(G^t{}_i\right) &= \delta\left(\frac{1}{\alpha}\left(\nabla_j K^j{}_i-\nabla_i K^j{}_j\right)\right) \\
&= \delta\left(\frac{1}{\alpha}\right)\left(\underbrace{0\text{th}}_{0}+\underbrace{1\text{st}}_{2\text{nd}}\right)+\frac{1}{\alpha^{(0)}}\left(\delta\nabla_j K^j{}_i-\underbrace{\delta\nabla_i K^j{}_j}_{3\delta\nabla_j K^j{}_i:(77)}\right) \\
&= \frac{1}{a}\left(-2\nabla_j\delta K^j{}_i\right) \\
&= -\frac{2}{a^2}\nabla_i(H\phi+\dot{\psi}) \qquad (92)
\end{aligned}
$$

The calculations of δR and $\delta R^i{}_j$ are alike, but we need

$$
\begin{aligned}
\delta\left(\frac{1}{\alpha}\nabla^i\nabla_j\alpha\right) &= \delta\left(\frac{1}{\alpha}\right)\left[\underbrace{\nabla^i\nabla_j\alpha^{(0)}}_{=0:\alpha^{(0)}=a(t)}+\nabla^i\nabla_j\underbrace{\delta\alpha}_{2\text{nd ord.}}\right] \\
&\quad+\frac{1}{\alpha}\left[\underbrace{\nabla^i\nabla_j\alpha^{(0)}}_{0}+\nabla^{(0)i}\nabla_j^{(0)}\delta\alpha\right] \\
&= \frac{1}{\alpha}\nabla^{(0)i}\nabla_j^{(0)}\delta\alpha=\frac{1}{\alpha}\nabla^{(0)i}\nabla_j^{(0)}\phi\alpha \\
&= \frac{1}{a^2}\nabla_i\nabla_j\phi, \qquad (93)
\end{aligned}
$$

where we used $\nabla^{(0)i}=g^{il}\nabla_l^{(0)}=\frac{1}{a^2}\gamma^{il}\nabla_l^{(0)}=\frac{1}{a^2}\nabla^i$.

With the same kind of calculations as above, only a little more terms involved, we find

$$
R = R(g)+K^i{}_jK^j{}_i+K^i{}_iK^j{}_j-\frac{2}{\alpha}\left[\Delta\alpha+\dot{K}^s{}_s\right] \quad (60)
$$

$$
R^{(0)} = \frac{6}{a^2}\left[K+H^2+\dot{H}\right] \qquad (94)
$$

$$
\begin{aligned}
\delta R &= \frac{2}{a^2}\left[(6K+2\Delta)\psi - \Delta\phi\right] \\
&\quad -\frac{6}{a^2}\left[H(\dot\phi + 3\dot\psi) + 2H(\dot H + H^2)\phi + \ddot\psi\right] \qquad (95)\\
R^i{}_j &= R^i{}_j(g) + K^s{}_s K^i{}_j - 2K^i{}_s K^s{}_j - \frac{1}{\alpha}\left[\nabla^i\nabla_j\alpha + g^{is}\dot K_{js}\right] \qquad (59)\\
R^{(0)i}_j &= \frac{1}{a^2}(2K + 2H^2 + \dot H)\delta^i{}_j \qquad (96)\\
\delta R^i{}_j &= \frac{\delta^i{}_j}{a^2}\left[4K+\Delta\right]\psi + \frac{\nabla^i\nabla_j}{a^2}\left[\psi - \phi\right] \\
&\quad -\frac{1}{a^2}\left[H(\dot\phi + 5\dot\psi) + 2(\dot H + 2H)^2\phi + \ddot\psi\right]\delta^i{}_j
\end{aligned}
$$

where the operators work on Σ.

3.4.3 Equations For $\delta G^\mu{}_\nu$

Finally, we compose the perturbations of the Einstein tensor like

$$
\begin{aligned}
\delta G^i{}_j &= \delta R^i{}_j - \frac{1}{2}\delta^i{}_j \delta R \\
&= -\frac{2K\psi}{a^2}\delta^i{}_j - (\Delta\psi - \Delta\phi)\frac{\delta^i{}_j}{a^2} + \frac{\nabla^i\nabla_j}{a^2}(\psi - \phi) \\
&\quad + \frac{2}{a^2}\left[H(\dot\phi + 2\dot\psi) + (2\dot H + H^2)\phi + \ddot\psi)\right]\delta^i{}_j .
\end{aligned}
$$

This is only a short calculation with the quantities from the last section.

Using the transformation

$$
\phi \longrightarrow \Phi, \qquad \psi \longrightarrow \Psi,
$$

with the Bardeen potentials (51) and (52), we find the gauge invariant expressions

$$
\begin{aligned}
\delta G^t{}_t &= \frac{2}{a^2}\left[3H(H\Phi + \dot\Psi) - (3K+\Delta)\Psi\right] \qquad (97)\\
\delta G^t{}_i &= -\frac{2}{a^2}\nabla_i(H\Phi + \dot\Psi) \qquad (98)\\
\delta G^i{}_j &= -\frac{2K\Psi}{a^2}\delta^i{}_j - (\Delta\Psi - \Delta\phi)\frac{\delta^i{}_j}{a^2} + \frac{\nabla^i\nabla_j}{a^2}(\Psi - \Phi) \\
&\quad + \frac{2}{a^2}\left[H(\dot\Phi + 2\dot\Psi) + (2\dot H + H^2)\Phi + \ddot\Psi)\right]\delta^i{}_j . \qquad (99)
\end{aligned}
$$

3.5 Perturbations Of The Stress-Energy Tensor

Until now, we only cared about the left side of Einstein's equations (156). Now we turn to the right side and ask ourselves what perturbations of $T_{\mu\nu}$ with a Robertson-Walker background must look like. Once again we find that four potentials parameterize $T_{\mu\nu}$:

scalar quantities	δT_{tt}	density fluctuations
vector quantities	δT_{ti}	$\sim \nabla_i$(scalar function) fluctuations in four-velocity
tensors	$tr(\delta T_{ij})$ δT_{ij} traceless	isotropic pressure fluctuations anisotropic pressure fluctuations

Calculating $(\delta T)^{\mu}{}_{\nu}$ is not that hard when we consider the four-velocity $u^\mu := (u^0, v)$ to be a normalized eigenvector of $T^{\nu}{}_{\mu}$ with eigenvalue $-\rho$:

$$\begin{aligned} g_{\mu\nu}u^\mu u^\nu &= -1 && (100)\\ T_\mu{}^\nu u^\mu &= -\rho u^\nu. && (101)\end{aligned}$$

The second condition is plausible in the following way: According to (185), $T^{(0)}_{\mu\nu}$ has only diagonal entries for a perfect fluid, which we will restrict ourselves to in dealing with a Robertson-Walker universe.[18] Then we see from (101) in comparison with (185) that the spacial components of the four-velocity to 0th order become $u^{(0)i} = 0$.

To 0th order, (100) becomes

$$\begin{aligned} g^{(0)}_{tt}u^{(0)t}u^{(0)t} &= -1 = -a^2u^{(0)t}u^{(0)t}\\ \Longrightarrow u^{(0)t} &= \frac{1}{a}, \quad u^{(0)}_t = -a. && (102)\end{aligned}$$

From the form (185) of $T_{\mu\nu}$ for a perfect fluid, we see that

$$T^{(0)t}_t = -\rho, \quad T^{(0)t}_i = 0, \quad T^{(0)i}_j = P\delta^i{}_j. \qquad (103)$$

When we calculate the perturbations $\delta T^\mu{}_\nu$, we will need the perturbations of u^μ also. They follow from (100):

$$\begin{aligned} \delta g_{\mu\nu}u^{(0)\mu}u^{(0)\nu} + 2g^{(0)}_{\mu\nu}\delta u^\mu u^{(0)\nu} &= 0\\ \Longrightarrow \delta g_{tt}u^{(0)t}u^{(0)t} + 2g^{(0)}_{tt}\delta u^t u^{(0)t} &= 0\end{aligned}$$

$$\begin{aligned} \Longrightarrow \delta u^t &= \frac{\delta g_{tt}(u^{(0)t})^2}{-2g^{(0)}_{tt}u^{(0)t}} = \frac{-2a^2\phi\frac{1}{a}}{2a^2} = -\frac{\phi}{a}\\ \Longrightarrow \delta u^t &= -\frac{\phi}{a}, \qquad \delta u_t = -a\phi.\end{aligned}$$

[18] A comprehensive treatment of perfect fluids in connection with the stress-energy tensor is given in chapter 12 of [3].

Now we can get the perturbations of the time and space-time components of $T_{\mu\nu}$ from (101)

$$\begin{aligned}
& \delta(T_\mu{}^\nu u^\mu + \rho u^\nu) = 0 \\
\Longleftrightarrow\quad & \delta T_\mu{}^\nu u^\mu + (T^{(0)})_\mu{}^\nu \delta u^\mu + \delta\rho u^\nu + \rho\delta u^\nu) = 0 \\
\Longleftrightarrow\quad & \delta T_\mu{}^\nu \delta^\mu{}_t u^t + \delta\rho\delta^\nu{}_\mu \delta^\mu{}_t u^t + \left((T^{(0)})_\mu{}^\nu + \rho\delta^\nu{}_\mu\right)\delta u^\mu = 0 \\
\Longleftrightarrow\quad & (\delta T_t{}^\nu + \delta\rho\delta^\nu{}_t)\frac{1}{a} + \left((T^{(0)})_\mu{}^\nu + \rho\delta^\nu{}_\mu\right)\delta u^\mu = 0,
\end{aligned}$$

which yields

$$\begin{aligned}
\delta T_t{}^t &= -\delta\rho \\
\delta T_t{}^i &= -a(P+\rho)\delta u^i.
\end{aligned}$$

Whereas we find also

$$\begin{aligned}
\delta T^t_j &= \delta\left(g_{j\nu} g^{t\mu} T^\nu{}_\mu\right) \\
&= a(\rho+P)\left[\gamma_{ij}\delta u^i - \frac{1}{a}B_{|j}\right]. \qquad (104)
\end{aligned}$$

We stated above that the four velocity contains the vector v which vanished to 0th order. But of course it appears in δu^i, and with this fact we sum up the results of this chapter:

$$\begin{aligned}
\delta u^t &= -\frac{1}{a}\phi && (105) \\
\delta u^i &:= \frac{1}{a}\gamma^{ij} v_{|j} \qquad \Longrightarrow \delta u_i = a(v-B)_{|i} && (106) \\
\delta T^t{}_t &= -\delta\rho =: -\rho\cdot\delta && (107) \\
\delta T^t{}_i &= (\rho+P)(v-B)_{|i} && (108) \\
\delta T^i{}_t &= -(\rho+P)\gamma^{ij} v_{|j} && (109) \\
\delta T^i{}_j &:= \delta\rho\delta^i{}_j + P\Pi^i{}_j && (110)
\end{aligned}$$

Don't let the new δ from (107) confuse you, it is called *density contrast* and just a quantity defined to help us write equations simpler, and regarding the new function Π, we will later see what it is good for. For now we only state its definition

$$\Pi^i{}_j := \left[\nabla^i\nabla_j - \delta^i{}_j \frac{1}{3}\Delta\right]\Pi. \qquad (111)$$

We now have parameterized $\delta T^\nu{}_\mu$ by $\frac{\delta\rho}{\rho}, v, \delta P$, and Π.

3.6 Gauge Invariant Matter Quantities

The perturbations of the Einstein tensor are formulated in a gauge invariant manner in (97) to (99). We are now formulating the perturbation equations for the $\delta T^{\nu}{}_{\mu}$ in a gauge invariant manner also, starting by calculating the gauge transformations

$$\begin{aligned}
\delta T^{t}{}_{t} &\longrightarrow \delta T^{t}{}_{t} + (L_X T)^{t}{}_{t} \\
\delta T^{i}{}_{j} &\longrightarrow \delta T^{i}{}_{j} + (L_X T)^{i}{}_{j} \\
\delta T^{t}{}_{i} &\longrightarrow \delta T^{t}{}_{i} + (L_X T)^{t}{}_{i}
\end{aligned}$$

where we find the Lie derivatives in the same way as in (43):

$$\begin{aligned}
(L_X T)^{t}{}_{t} &= -\grave{T}\dot{\rho} \\
(L_X T)^{i}{}_{j} &= \grave{T}\dot{P}\delta^{i}{}_{j} \\
(L_X T)^{t}{}_{i} &= -\rho\grave{T}_{|i} - P\delta^{i}{}_{j}\grave{T}_{|j} = -(\rho+P)\grave{T}_{|i}.
\end{aligned}$$

Here, $\grave{T}$ denotes the gauge freedom T from (53), to distinguish it from the stress-energy tensor. Comparing the last few equations, we find the transformation of the four potentials describing the perturbations of $T^{\nu}{}_{\mu}$ to be

$$\begin{aligned}
\delta\rho &\longrightarrow \delta\rho + \grave{T}\dot{\rho} && (112)\\
\delta P &\longrightarrow \delta P + \grave{T}\dot{P} && (113)\\
(v-B) &\longrightarrow (v - B - \grave{T}) && (114)\\
\Pi &\longrightarrow \Pi. && (115)
\end{aligned}$$

The task is now to find gauge invariant quantities for (112) to (114), since only Π is gauge invariant. Note that in all three cases we have to consider, there is always the function $\grave{T}$ appearing after the transformation with a derivation of the unperturbed quantity which was put in on the left side. Looking at (49) and (50), we can compose these two functions to $\dot{E}+B$, which transforms like

$$\dot{E} + B \longrightarrow \dot{E} + B + \grave{T}$$

in the notation we use now.

Of course, this solves our problem: Adding $\dot{E}+B$ times the negative of the factor that appears after the gauge transformations (112) to (114) to any unperturbed quantity gives the three new gauge invariant potentials

$$\begin{aligned}
V &:= (v-B) + (\dot{E}+B) = v + \dot{E} && (116)\\
\triangle_s &:= \frac{1}{\rho}\Big(\delta\rho - \dot{\rho}(\dot{E}+B)\Big) && (117)\\
\Gamma &:= \frac{1}{P}\Big(\delta P - \dot{P}(\dot{E}+B)\Big). && (118)
\end{aligned}$$

V is called the *gauge invariant velocity*, $\triangle_s$ is called the *gauge invariant density contrast*.

Directly from Einstein's equations, one finds[19]

$$\dot{\rho} = -3H(\rho + P). \tag{119}$$

We set

$$\begin{aligned} w &:= \frac{P}{\rho} && (120) \\ c_s^2 &:= \frac{\dot{P}}{\dot{\rho}}, && (121) \end{aligned}$$

and find from (117)

$$\triangle_s = \delta + 3H(1+w)(\dot{E} + B). \tag{122}$$

Note that we find two of the gauge invariant potentials from matter quantities alone: The first is the anisotropic part of $\delta T^i{}_j$ parameterized by Π. For the second, we use (112) and (113) to find an expression that must be equal to Γ, since we used all transformations (112) to (115) to find the gauge invariant potentials except for (113):

$$\Gamma = \frac{1}{P}\left(\delta P - \frac{\dot{P}}{\dot{\rho}}\delta\rho\right). \tag{123}$$

It follows from $\Gamma = 0$ that $\frac{\delta P}{\delta\rho} = \frac{\dot{P}}{\dot{\rho}} = 0$. In this way, the *adiabatic perturbation* has gauge invariance meaning, and in general, Γ parametrizes the entropy production of a perturbation.

We state another way to write the density contrast, which we will use afterwards:

$$\begin{aligned} \triangle_g &:= \triangle_s - 3(1+w)\Psi && (124) \\ \triangle &:= \triangle_s - 3(1+w)HV. && (125) \end{aligned}$$

It will always be clear from context that $\triangle$ cannot be the Laplace operator.

With all gauge invariant quantities $\triangle_s, V, \Gamma$, and Π, we write the perturbations of $T^\nu{}_\mu$ in gauge invariant manner, using (107) to (110):

$$\begin{aligned} \delta T^t{}_t &= -\rho \triangle_s && (126) \\ \delta T^t{}_i &= \rho(1+w)V_{|i} && (127) \\ \delta T^i{}_j &= \rho\left(w\Gamma + c_s^2 \triangle_s + w\Pi^i{}_j\right). && (128) \end{aligned}$$

These equations are easily found with a little algebra, except for $\delta T^i{}_j$, for which we need both equations for Γ.

[19] As is shown in [5], p. 435.

We arrive at the gauge invariant expressions for Einstein's equations on both sides:

$$
\begin{aligned}
\delta G^t{}_t &= 8\pi\delta T^t{}_t : \\
3H(H\Phi + \dot{\Psi}) - (3K + \Delta)\Psi &= -\frac{8\pi}{2}\rho a^2 \triangle_s \qquad (129)
\end{aligned}
$$

$$
\begin{aligned}
\delta G^t{}_i &= 8\pi\delta T^t{}_i : \\
H\Phi + \dot{\Psi} &= -\frac{8\pi}{2}\rho a^2(1+w)V \qquad (130)
\end{aligned}
$$

$$
\begin{aligned}
\delta G^i{}_j &= 8\pi\delta T^i{}_j : \\
\left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\Delta\right)(\Phi + \Psi) &= -8\pi\rho a^2 w\Pi_{ij}, \qquad \text{and} \qquad (131) \\
\frac{1}{3}\Delta(\Phi - \Psi) + (H^2 + 2\dot{H})\Phi &- K\Psi + \left[\ddot{\Psi} + H\dot{\Phi} + 2H\dot{\Psi}\right] \\
&= \frac{8\pi}{2}\rho a^2\left[w\Gamma + c_s^2\triangle_s\right]. \qquad (132)
\end{aligned}
$$

The last two equations correspond to the traceless part and the trace of $\delta G^i{}_j$.

From (131), we find the difference of the gauge invariant Bardeen potentials to be

$$\Psi - \Phi = 8\pi\rho a^2 w\Pi \qquad (133)$$

what must correspond to the *anisotropic pressure.*

Insert (130) into (129) to find

$$(\Delta + 3K)\,\Psi = \frac{8\pi}{2}\rho a^2 \triangle . \qquad (134)$$

The previous two equations yield expressions for the gauge invariant Bardeen potentials in quantities of matter

$$
\begin{aligned}
\Psi &= \frac{8\pi}{2}\rho a^2 \triangle [\Delta + 3K]^{-1} \qquad (135) \\
\Phi &= \frac{8\pi}{2}\rho a^2 \left[(\Delta + 3K)^{-1}\triangle - 2w\Pi\right], \qquad (136)
\end{aligned}
$$

where we have to remember that $\triangle$ is an expression for the density contrast, whereas Δ stands for the Laplace operator.

4 Application Of The Perturbation Equations

The perturbed Einstein equations alone do not impress us very much. We need to apply them to real problems to see what they are useful to. Remember from the introduction that we wanted to find out whether galaxies can be the end products of density fluctuations - this chapter will give a hint (although there is more to the answer than we can include here).

First, we calculate the evolution of the gauge invariant density contrast and the gauge invariant velocity through application of the stress-energy conservation. Afterwards, we find the sources of density flucuations, rearranging the fundamental equations we have found in the first section. At the end, we will state a solution for a universe filled only with dust and radiation.

4.1 Evolution Of Density Contrast And Velocity

With all we have achieved until now, we can without great effort calculate the evolution of $\triangle$ and V. These quantities bear information about how our universe evolves over time. We will get the evolution equations for $\triangle$ and V from the conservation of the stress-energy tensor

$$\delta\left[T^{\mu}_{\ \nu};\mu\right] = 0,$$

for $\nu = i$ and $\nu = t$, respectively. Here, the semicolon ; denotes covariant differentiation, which is defined such that

$$A^{i}_{\ j};k = A^{i}_{\ j},k + \Gamma^{i}_{\ kl}A^{l}_{\ j} - \Gamma^{l}_{\ kj}A^{i}_{\ l}.$$

We are going to use

$$T^{t}_{\ t} = -\rho, \qquad T^{i}_{\ j} = P\delta^{i}_{\ j}, \qquad T^{t}_{\ i} = 0,$$

and the Christoffel symbols calculated in $g^{(0)}$ like in the appendix,

$$\Gamma^{t}_{tt} = H, \quad \Gamma^{t}_{ti} = \Gamma^{i}_{\ tt} = \Gamma^{i}_{\ ii} = 0, \quad \Gamma^{t}_{\ tj} = H\gamma_{ij}, \quad \Gamma^{i}_{\ tj} = H\delta^{i}_{\ j}.$$

For example,

$$\begin{aligned} \Gamma^{t}_{\ tt} &= \frac{1}{2}g^{tl}\left[g_{tl,t} - g_{lt,t} + g_{tt,l}\right] \\ &= \frac{1}{2}g^{tt}g_{tt,t} = \frac{1}{2}\frac{1}{a^2}\gamma^{tt}2\dot{a}a\gamma_{tt} \\ &= H. \end{aligned}$$

Then, a short calculation shows that

$$\begin{aligned} \delta\left[T^{\mu}_{\ t};\mu\right] &= (\delta T^{i}_{\ t})_{,i} + \delta\dot{T}^{t}_{\ t} - H\delta T^{i}_{\ i} + 3H\delta T^{t}_{\ t} - (\rho + P)\delta\Gamma^{i}_{\ it} \\ \delta\left[T^{\mu}_{\ i};\mu\right] &= (\delta T^{j}_{\ i})_{,j} + \delta\dot{T}^{t}_{\ i} + 3H\delta T^{t}_{\ i} - H\gamma_{ij}\delta T^{j}_{\ t} + (\rho + P)\delta\Gamma^{t}_{\ ti} \end{aligned}$$

Everything in these equations is known except for the $\delta\Gamma$s, which one gets through direct computation

$$\begin{aligned}
\delta\Gamma^{i}{}_{it} &= \delta\left(\frac{1}{2}g^{ij}\left[g_{ij,t}+g_{tj,i}-g_{it,j}\right]\right)\\
&= \delta\left(\frac{1}{2}g^{ij}\dot{g}_{ij}\right)\\
&= \frac{1}{2}\left(\delta g^{ij}\dot{g}^{(0)}_{ij}+g^{(0)ij}\delta\dot{g}_{ij}\right)\\
&= \frac{1}{2}\left(\frac{2}{a^2}\left[\psi\gamma^{ij}-\nabla^i\nabla_j\gamma_{ij}E\right]\right)\cdot(2a\dot{a}\gamma_{ij})\\
&\quad +\frac{1}{2}\frac{1}{a^2}\gamma^{ij}\left(-2a^2\left[\psi\gamma_{ij}-\nabla_i\nabla_j E\right]\right)^{\cdot}\\
&= 2H\left(\psi\gamma^{ij}\gamma_{ij}-\nabla^i\nabla_j\gamma_{ij}E\right)-2H\left[\psi\gamma^{ij}\gamma_{ij}-\nabla_i\nabla_j\gamma^{ij}E\right]\\
&\quad -\gamma^{ij}\left[\dot{\psi}\gamma_{ij}-\nabla_i\nabla_j\dot{E}\right]\\
&= -3\dot{\psi}+\Delta\dot{E}
\end{aligned}$$

$$\delta\Gamma^{t}{}_{ti} = \phi_{|i}-HB_{|i}.$$

These are all necessary quantities to calculate the perturbation of the stress-energy conservation:

$$\begin{aligned}
\delta\left[T^{\mu}{}_{t};\mu\right] &= (\delta T^{i}{}_{t})_{,i}+\delta\dot{T}^{t}{}_{t}-H\delta T^{i}{}_{i}+3H\delta T^{t}{}_{t}-(\rho+P)\delta\Gamma^{i}{}_{it}\\
&= \left(-(\rho+P)\gamma^{ij}v_{|j}\right)_{|i}-(\delta\cdot\rho)^{\cdot}-H(\delta P\delta^{i}{}_{i}+P\Pi^{i}{}_{i})\\
&\quad -3H\rho\cdot\delta+(\rho+P)(3\dot{\psi}-\Delta\dot{E})\\
&= -(\delta\cdot\rho)^{\cdot}-3H(\rho\cdot\delta+\delta P)\\
&\quad -(\rho+P)\left[\gamma^{ij}v_{|j|i}-3\dot{\psi}+\Delta\dot{E}\right]-PH\Pi^{i}{}_{i} \qquad (137)\\
&= -(\delta\cdot\rho)^{\cdot}-3H(\delta P-\rho\cdot\delta)\\
&\quad +(\rho+P)\left[\Delta(v+\dot{E})-3\dot{\psi}\right]
\end{aligned}$$

$$\begin{aligned}
\delta\left[T^{\mu}{}_{i};\mu\right] &= \delta P_{|i}+(\rho+P)^{\cdot}(v-B)_{|i}+(P+\rho)\left[(v-B)^{\cdot}\right.\\
&\quad \left.+4H(v-B)+\phi\right]_{|i}+p\Pi^{j}_{i|j}. \qquad (138)
\end{aligned}$$

It follows when we integrate the stress-energy conservation and use $\Pi^{j}_{i|j}=\left[\frac{2}{3}(\Delta+3K)\right]_{|i}$ (this you can get from writing out $\Pi^{|j}_{|j|i}$ in terms of $\Pi^{|j}_{|i|j}+R_{ij}\Pi^{|j}$ and use the appropriate definitions):

$$0 = \dot{\delta}-3\psi(1+w)+\delta\underbrace{\left[3H+\frac{\dot{\rho}}{\rho}\right]}_{-3Hw}+3Hw\frac{\delta P}{P}$$

$$
\begin{aligned}
& +(1+w)\Delta(v+\dot{E}) \\
0 \quad = \quad & \delta P + (P+\rho)^{\cdot}(v-B) + \rho(1+w)\left[(v-B)^{\cdot}\right. \\
& \left. +4H(v-B)+\phi\right] + p\frac{2}{3}(\Delta+3K)\Pi,
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
0 \quad &= \quad \dot{\delta} + 3Hw\left[\frac{\delta P}{P} - \delta\right] + (1+w)\left[\Delta(v+\dot{E}) - 3\dot{\psi}\right] && (139) \\
0 \quad &= \quad w\frac{\delta P}{P} + \frac{2}{3}w(\Delta+3K)\Pi \\
& \quad\; +(1+w)\left[(v-B)^{\cdot} + H(1-3c^{s})(v-B)+\phi\right], && (140)
\end{aligned}
$$

but these equations are not gauge invariant yet. We have to insert the gauge invariant quantities we know so far,

$$
\begin{aligned}
\delta &= \triangle_g + 3(1+w)\psi \\
\phi &= \Phi + \frac{1}{a}\left(a(\dot{E}+B)\right)^{\cdot} \\
\psi &= \Psi - H(\dot{E}+B) \\
\frac{\delta P}{P} &= \Gamma + \frac{c_s^2}{w}\delta \\
V &= v + \dot{E},
\end{aligned}
$$

and use the relation

$$
\begin{aligned}
\dot{\delta} &= \dot{\triangle}g + 3(1+w)\dot{\psi} + 3\dot{w}\psi \\
&= \dot{\triangle}g + 3(1+w)\dot{\psi} + 3\psi\left[3H(1+w)(w-c_s^2)\right], && (141)
\end{aligned}
$$

where

$$
\dot{w} = (c_s^2 - w)\frac{\dot{\rho}}{\rho} = 3H(1+w)(w-c_s^2).
$$

Now we can write (139) and (140) in gauge invariant form:

$$
0 = \dot{\triangle}_g + 3(c_s^2 - w)H\,\triangle_g + (1+w)\Delta V + 3Hw\Gamma \qquad (142)
$$

$$
\begin{aligned}
0 \quad = \quad & (1+w)\left[\dot{V} + HV(1-3c_s^2) + \Phi + 3c_s^2\Psi\right] + w\Gamma \\
& + c_s^2\,\triangle_g + \frac{2}{3}w(\Delta+3K)\Pi. && (143)
\end{aligned}
$$

Collecting all c_s^2-terms and inserting Φ from (136) yields

$$
\begin{aligned}
(1+w)\left[\dot{V}+HV\right] &= \left[\frac{8\pi}{2}\rho a^2 2 - \frac{2}{3}(\Delta+3K)\right] w\Pi - w\Gamma \\
&\quad - \left[\frac{8\pi}{2}\rho a^2(\Delta+3K)^{\cdot} + c_s^2\right] \triangle\,. \qquad (144)
\end{aligned}
$$

By definition,

$$\triangle := \triangle_g + 3(1+w)\left[\Psi - HV\right],$$

and from that,

$$\dot{\triangle} = \dot{\triangle}_g + 3(1+w)\left[\dot{\Psi} - H\dot{V}\right] - 3V\left[(1+w)\dot{H} + H\dot{w}\right].$$

Here we insert (141) for $\triangle_g$, bring the perturbed ${G^t}_i$ (130) into the form

$$\dot{\Psi} + H\Phi = \frac{8\pi}{2}\rho a^2\left[H2w\Pi - (1+w)V\right],$$

to insert it for $\dot{\Psi}$, take (144) for $\dot{V}$ and the background for $\dot{H}$ and $\dot{w}$:

$$
\begin{aligned}
\dot{H} &= H^2 + K - \frac{1}{2}8\pi\rho a^2(1+w) \\
\dot{w} &= 3H(1+w)(w-c_s)^2.
\end{aligned}
$$

We finally find

$$\dot{\triangle} - 3Hw\triangle = (\Delta+3K)\left[2wH\Pi - (1+w)V\right]. \qquad (145)$$

Equations (144) and (145) are the fundamental equations for the evolution of $\triangle$ and V. They are two coupled ordinary differential equations for $\triangle$ and V.

4.2 Sources Of Density Fluctuations

If we want to explain the existence of galaxies, we have to understand the evolution of the density contrast $\triangle$ and of the velocity V. We have achieved this with the last section, but we do not see yet where the density fluctuations stem from.

To discover that, we consider a decomposition of our gauge invariant quantities $\triangle, V, \Pi$, and Γ in modes, expanded in $Y^{(k)}$. We remember that according to (108) and (116), V parameterizes a gradient and according to (111), Π parametrizes a twofold covariant derivative.

Thus, for such a decomposition,

$$\triangle \longrightarrow \triangle_{(k)}, \qquad \Gamma \longrightarrow \Gamma_{(k)}, \qquad V \longrightarrow -\frac{1}{k}V_{(k)}, \qquad \Pi \longrightarrow \frac{1}{k^2}\Pi_{(k)}.$$

(Consider a fourier transform to figure out the k-factors, k stands for the wave number.) With the same argument, for the Laplace operator,

$$\Delta \longrightarrow -k^2.$$

Our fundamental equations for the evolution of $\triangle$ (145) and V (144) then become:

$$\begin{aligned}
\dot{\triangle} - 3wH\triangle &= \left[\frac{3K}{k^2} - 1\right]\left[(1+w)kV + 2wH\Pi\right] && (146)\\
\dot{V} + HV &= -k\left[\frac{\frac{2}{3}}{1+w}\left(1 - \frac{3K}{k^2}\right) + \frac{8\pi\rho a^2}{k^2}\right] w\Pi + k\frac{w}{1+w}\Gamma \\
&\quad + k\left[\frac{c_s^2}{1+w} - \frac{8\pi\rho a^2}{2k^2}\left(1 - \frac{3K}{k^2}\right)^{-1}\right]\triangle, && (147)
\end{aligned}$$

where K denotes the curvature and k the wave number. To illustrate this situation, we must remember that what we are looking at are still stress-energy conservation equations, which we brought in a form where the evolution of the density contrast $\triangle$ and the velocity V becomes obvious. It is intuitive to imagine the density fluctuations described by these equations to spread as waves with wavelength λ and wave number k.

With $\frac{\dot{\rho}}{\rho} = -3H(1+w)$[20], we can write the left side of (146) as

$$\dot{\triangle} - 3wH\triangle = \frac{(\triangle a^3 \rho)^{\cdot}}{a^3\rho}. \qquad (148)$$

This shows that $-3wH\triangle$ is the adiabatic change of the density contrast, caused by cosmic expansion.

Through elimination of V in (146) and (147) one can get (after a fairly complicated but only algebraic calculation)

$$\begin{aligned}
\ddot{\triangle} &+ \left[\triangle + 3(c_s^2 - 2w)\right] H\dot{\triangle} + 3\alpha_1\triangle = S && (149)\\
S &:= \left(1 - \frac{3K}{k^2}\right)\left[2\alpha_2\Pi - 2Hw\dot{\Pi} - wk^2\Gamma\right], && (150)
\end{aligned}$$

with the background functions

$$\begin{aligned}
\alpha_1 &:= H^2\left[\frac{3}{2}w^2 - 4w + 3c_s^2 - \frac{1}{2}\right] + \frac{1}{2}\left(3w^2 - 1\right)K + \frac{1}{3}\left(k^2 - 3K\right)c_s^2\\
\alpha_2 &:= H^2\left[3w^2 - 2w + 3c_s^2\right] + w(3w+2)K + (k^2 - 3K)c_s^2.
\end{aligned}$$

The left side of (149) is determined by the density contrast in second order and on the right side we have Γ and Π. From (123) and (133) and their remarks we can conclude that *the sources of density fluctuations are entropy perturbations* Γ *and anisotropic pressure perturbations* Π.[21]

[20] A direct consequence from Einstein's equations in a Robertson-Walker universe, see [5], p. 435.

[21] [1], p. 47 and following pages, gives an account on how the effects of Γ and Π can be estimated and how the exact solutions can only be found if we treat matter as a perfect fluid.

4.3 Solution For A Universe Filled With Dust and Radiation

Finally, for ease of use, we document the solution for a dust-radiation universe. We exactly follow Kodama and Sasaki in [1].

For the following calculations, we introduce a new variable $\xi \sim e^a$ in place of the conformal time t, defined by

$$\dot{\xi} =: \dot{a}\xi = k\frac{\dot{a}}{a}\frac{a}{k}\xi = kHl\xi, \tag{151}$$

where $l := \frac{a}{k} \sim a \cdot \lambda$. When we write Hl as

$$Hl = \frac{\frac{a}{k}}{\frac{1}{H}} = \frac{\text{reduced wavelength of perturbation}}{\text{radius of Hubble horizon}}, \tag{152}$$

we see the sense of Hl in (151): We are interested in how large perturbations are compared to the hubble horizon H^{-1} and how fast they grow with time.

ξ shall be defined such that $\xi = 1$ when the perturbation enters the Hubble horizon:

$$\xi(Hl = 1) := 1. \tag{153}$$

We set the curvature $K = 0$ and express (149) in terms of ξ:

$$\frac{d^2\triangle}{d\xi^2} - \frac{\mu}{\xi}\frac{d\triangle}{d\xi} + \left[\frac{c_s^2}{f^2} - \frac{2+\nu}{\xi^2}\right]\triangle = S \tag{154}$$

$$S = -\frac{w}{f^2}\left[\Gamma - \frac{2}{3}\Pi\right] + \frac{2}{\xi}\left(3w^2 + 3c_s^2 - 2w\right)\Pi - \frac{2w}{\xi}\frac{d\Pi}{d\xi}$$

with the background quantities

$$\begin{aligned}
\mu &:= -\frac{5}{2}(1-3w) + 1 - 3c_s^2 \\
\nu &:= -\frac{1}{2}(1-3w)(7-3w) + 3(1-3c_s^2) \\
f &:= Hl\xi.
\end{aligned}$$

If we only consider density fluctuations in a universe that is only filled with dust, meaning pressure-free particles, and radiation, we must have[22]

$$\begin{aligned}
\rho &= \frac{1}{2}\xi^{-4} + \frac{1}{2}\xi^{-3} \\
P &= \frac{1}{6}\xi^{-4},
\end{aligned}$$

and therefore, with $z := 1 + \xi$,

[22] [1], p. 70.

$$\begin{aligned} w &= \frac{P}{\rho} = \frac{1}{3z} \\ c_s^2 &= \frac{\dot{P}}{\dot{\rho}} = \frac{4}{3(1+3z)}. \end{aligned}$$

For adiabatic perturbations is $S = 0$, and if the perturbations extend far beyond the horizon, $\frac{c_s}{f} \ll 1$,

$$\frac{d^2\triangle}{d\xi^2} - \frac{\mu}{\xi}\frac{d\triangle}{d\xi} + \left[-\frac{2+\nu}{\xi^2}\right]\triangle = 0$$

In this form, we can solve (154) analytically. With the previous equations for μ, ν, w, and c_s^2, one finds

$$\begin{aligned} & \frac{d^2\triangle}{dz^2} + \left(\frac{5}{2z} - \frac{3}{3z+1}\right)\frac{d\triangle}{dz} \\ + \left(\frac{\frac{27}{4}}{3z+1} - \frac{3}{z} + \frac{1}{2z^2} + \frac{\frac{3}{4}}{z-1} - \frac{2}{(z-1)^2}\right)\triangle &= 0. \end{aligned} \tag{155}$$

To find the general solution of this differential equation, Kodama and Sasaki make the Ansatz[23]

$$\triangle = \frac{u(z)}{\sqrt{z}(z-1)},$$

obviously to get rid of the factor that comes with $\triangle$, which simplifies the equation a lot. Now, we can write (155) as

$$\frac{d^2u}{dz^2} + \left[\frac{3}{2z} - \frac{2}{z-1} - \frac{3}{3z+1}\right]\frac{du}{dz} = 0,$$

which has the general solution

$$u(z) = c_1 \frac{1}{\sqrt{z}}\left(z^3 - \frac{25}{9}z^2 + \frac{5}{3}z - \frac{5}{3}\right) + c_2.$$

c_1 and c_2 are arbitrary integration constants. This means for $\triangle$:

$$\begin{aligned} \triangle(z) &= AU_{\nearrow}(z) + BU_{\searrow}(z) \\ U_{\nearrow} &= \left[c + \frac{1}{\sqrt{z}}\left(z^3 - \frac{25}{9}z^2 + \frac{5}{3}z - \frac{5}{3}\right)\right]U_{\searrow} \\ U_{\searrow} &= \frac{1}{\sqrt{z}(z-1)}. \end{aligned}$$

A and B are arbitrary constants, and c has to be determined.

In the matter dominated era, where $z \gg 1$, obviously $U_{\searrow} \sim \xi^{-\frac{3}{2}}$ and $U_{\nearrow} \sim \xi$. These are the decaying and growing modes in the matter dominated era.

[23] [1], p. 70.

A Einstein's Equations In The Robertson–Walker Metric

In this appendix, we manually calculate Einstein's equations for a Robertson–Walker universe. The sense of doing so is to get insight into what the Robertson-Walker metric is all about, how the Γs work, and what Einstein's equations looks like in this metric: It yields a set of coupled differential equations for $a(t), \rho$, and P, which then describe a homogeneous, isotropic universe. Remember that this is one of only few exact solutions of Einstein's equations.

Using a *2+2 formulation* that separates space and time from the angular coordinates is a rather elegant way of dealing with the Christoffel symbols appearing in the Riemann Tensor. This in itself is worth reading if one is not familiar with that idea.

A.1 Aim And Plan

Our aim is to write Einstein's equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{156}$$

in the explicit form for a homogeneous, isotropic universe, which can be described using the Robertson-Walker metric introduced below.

The left side consists of the *Einstein tensor*[24]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{157}$$

The *Ricci tensor* $R_{\mu\nu}$ is the contraction of the *Riemann tensor* $R^{\alpha}_{\mu\alpha\nu}$. The *Ricci Scalar* R is defined by $R = R^{\mu}{}_{\mu}$. Of course, we can raise and lower indices after we introduce a metric in the usual way (for example, $R^{\alpha}{}_{\beta} = g^{\alpha\lambda} R_{\lambda\beta}$).

After these remarks, we can write the Einsteintensor (157) as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} R_{\alpha\beta} \right). \tag{158}$$

We will use the *Robertson–Walker Metric* which is fit to describe a homogeneous, flat universe. In this metric, the *line element* becomes

$$ds^2 = -dt^2 + a^2 \frac{dr^2}{1 - Kr^2} + a^2 r^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right). \tag{159}$$

We used the notation $a = a(t)$ for the *cosmic scale factor* and K for the *constant curvature*. Obviously, we are working with the four coordinates t, r, ϑ and φ

According to (157) , the plan is now to calculate all the Riemann tensors to find the Ricci tensors and the Ricci scalar necessary for the left side of Einstein's equation (156).

[24] As Wald explaines in chapter 4.3 of [6].

Since we can in our case calculate the components of the Riemann tensor by[25]

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\mu\alpha} - \Gamma^{\alpha}_{\lambda\alpha}\Gamma^{\lambda}_{\mu\nu}, \tag{160}$$

we have to concentrate on a comfortable way of calculating the Christoffel symbols.[26]

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\mu,\nu} + g_{\delta\nu,\mu} - g_{\mu\nu,\delta} \right). \tag{161}$$

A.2 Explicit Form Of The Robertson–Walker Metric

By definition,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

and since there are no off–diagonal terms,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{tt} dt^2 + g_{rr} dr^2 + g_{\vartheta\vartheta} d\vartheta^2 + g_{\varphi\varphi} d\varphi^2. \tag{162}$$

The comparison of the coefficients of (162) and (159) yields

$$\mathbf{g}_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin\varphi^2 \end{bmatrix}, \tag{163}$$

with the inverse matrix

$$\mathbf{g}^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-Kr^2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2 r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2 r^2 \sin\varphi^2} \end{bmatrix}. \tag{164}$$

A.3 2+2 Formulation

If ones has to calculate Christoffel symbols (161) in the Robertson–Walker metric, a simple idea makes life a lot easier: Separate the t and $r$coordinate from the ϑ and φ parts. The Γs consist of derivations of the metric, and one knows e.g. that only the $g_{\varphi\varphi}$ depends on an angle, namely on ϑ. If we use small letters for t or r and capitals for the angular coordinates, the last sentence means

$$g_{\mu\nu,A} \begin{cases} \neq 0 & \text{if } \mu = \nu = \varphi \\ = 0 & \text{else} \end{cases}. \tag{165}$$

[25] See chapter 3 of [3] for a proof.

[26] The Christoffel symbols are introduced in a general context as components of a connection in chapter 3 of [6].

Such ideas consequently applied calculating the Γs and adding them to yield the Riemann tensor will make things rather easy to calculate by hand.

We write the Robertson–Walker metric (159) like

$$g = \underbrace{-dt^2 + a^2 \frac{dr^2}{1-Kr^2}}_{g_{tt}dt^2 + g_{rr}dr^2} + \underbrace{a^2 r^2 d\Omega^2}_{F^2(r,t)d\Omega^2},$$

where the elements of (163) appear and we absorb the factor a^2r^2 in a new function $F^2(r,t)$ and the angular parts of the metric in $d\Omega^2$.

So we can write the metric as

$$g = \widetilde{g}_{ij} dx^i dx^j + F^2 \widehat{g}_{AB} dx^A dx^B$$

using our convention for small and capital latin letters, $i = t, r; j = t, r; A = \vartheta, \varphi; B = \vartheta, \varphi$. The $\widetilde{g}$ part now only depends on t and r, while $\widehat{g}$ depends only on ϑ and φ.

A.4 Calculation Of The Γs

The 2+2 formulation leaves us with only six different cases of Christoffel symbols (161), namely

$$\Gamma^i{}_{jk}, \Gamma^i{}_{Aj}, \Gamma^i{}_{AB}, \Gamma^C{}_{AB}, \Gamma^C{}_{Ai}, \Gamma^C{}_{ij}. \tag{166}$$

Calculating and then inserting t, r, ϑ and φ into (166) gives an overview of all Christoffel symbols:

$$\begin{aligned}
\Gamma^i{}_{jk} &= \frac{1}{2} g^{il} \left(g_{lj,k} + g_{lk,j} - g_{jk,l} \right) && (167) \\
\Gamma^i{}_{Aj} &= \frac{1}{2} g^{il} \left(g_{lA,j} + g_{lj,A} - g_{Aj,l} \right) && (168) \\
\Gamma^i{}_{AB} &= \frac{1}{2} g^{il} \left(g_{lA,B} + g_{lB,A} - g_{AB,l} \right) && (169) \\
\Gamma^C{}_{AB} &= \frac{1}{2} g^{CD} \left(g_{DA,B} + g_{DB,A} - g_{AB,D} \right) && (170) \\
\Gamma^C{}_{Ai} &= \frac{1}{2} g^{CD} \left(g_{DA,i} + g_{Di,A} - g_{Ai,D} \right) && (171) \\
\Gamma^C{}_{ij} &= \frac{1}{2} g^{CD} \left(g_{Di,j} + g_{Dj,i} - g_{ij,D} \right) && (172)
\end{aligned}$$

A closer look at each case reveals the few calculations that really need to be carried out:

(167) yields two further cases, one from $i = t$ and one from $i = r$:

$$\begin{aligned}
\Gamma^t{}_{jk} &= \frac{1}{2} g^{tt} \left(g_{tj,k} + g_{tk,j} - g_{jk,t} \right) && (173) \\
\Gamma^r{}_{jk} &= \frac{1}{2} g^{rr} \left(g_{rj,k} + g_{rk,j} - g_{jk,r} \right). && (174)
\end{aligned}$$

(168) = 0, because it either contains non–diagonal entries of (163), which are 0, or a derivation of g_{lj} by an angle, which is also 0 because of (165).

(169) yields two further cases:

$$\begin{aligned}\Gamma^{t}{}_{AB} &= \frac{1}{2}g^{tt}\left(-g_{AB,t}\right)\\ \Gamma^{r}{}_{AB} &= \frac{1}{2}g^{rr}\left(-g_{AB,r}\right)\end{aligned}$$

with the above remarks.

(170) is seen to be

$$\Gamma^{C}{}_{AB} = \frac{1}{2}\frac{1}{F^2}\widehat{g}^{CD}\left(F^2\widehat{g}_{DA,B} + F^2\widehat{g}_{DB,A} - F^2\widehat{g}_{AB,D}\right) = \widehat{\Gamma}^{C}{}_{AB}$$

if one crosses F^2. The corresponding cases can be calculated easily: We only have to consider $g_{AB,\varphi}$, for we know that there can no other derivations be involved.

(171) yields two further cases:

$$\begin{aligned}\Gamma^{C}{}_{At} &= \frac{1}{2}g^{CD}\left(g_{DA,t}\right) = \frac{1}{2}\frac{1}{F^2}\widehat{g}^{CD}\left(F^2_{,t}\widehat{g}_{DA}\right)\\ &= \frac{1}{2}\frac{1}{a^2r^2}\widehat{g}^{CD}\left(2a\dot{a}r^2\widehat{g}_{DA}\right) = \frac{\dot{a}}{a}\widehat{g}^{CD}\widehat{g}_{DA} = \frac{\dot{a}}{a}\delta^{C}{}_{A}\\ \Gamma^{C}{}_{Ar} &= \frac{1}{2}g^{CD}\left(g_{DA,r}\right) = \frac{1}{2}\frac{1}{F^2}\widehat{g}^{CD}\left(F^2_{,r}\widehat{g}_{DA,r}\right) = \frac{1}{r}\delta^{C}{}_{A}\end{aligned}$$

From the definition of the Kronecker Delta, these two cases are $\neq 0$ only if $C = A$.

(172) = 0 for the same reasons as (168).

With these remarks on the six cases for Christoffel symbols (167) to (172), we can calculate the complete list of Christoffel symbols by hand and find:

$\Gamma^t{}_{tt} = 0$	$\Gamma^\vartheta_{\vartheta\vartheta} = 0$
$\Gamma^t{}_{tr} = 0$	$\Gamma^\vartheta_{\vartheta\varphi} = 0$
$\Gamma^t{}_{rt} = 0$	$\Gamma^\vartheta_{\varphi\vartheta} = 0$
$\Gamma^t{}_{rr} = \frac{a\dot{a}}{1-Kr^2}$	$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta$
$\Gamma^t_{\vartheta\vartheta} = a\dot{a}r^2$	$\Gamma^\vartheta_{\vartheta t} = \frac{\dot{a}}{a}$
$\Gamma^t_{\vartheta\varphi} = 0$	$\Gamma^\vartheta_{t\vartheta} = \frac{\dot{a}}{a}$
$\Gamma^t_{\varphi\vartheta} = 0$	$\Gamma^\vartheta_{\vartheta r} = \frac{1}{r}$
$\Gamma^t_{\varphi\varphi} = a\dot{a}r^2\sin^2\vartheta$	$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}$
$\Gamma^r{}_{rr} = \frac{Kr}{1-Kr^2}$	$\Gamma^\varphi_{\varphi\varphi} = 0$
$\Gamma^r{}_{tr} = \frac{\dot{a}}{a}$	$\Gamma^\varphi_{\varphi\vartheta} = \frac{\sin\vartheta}{\cos\vartheta}$
$\Gamma^r{}_{rt} = \frac{\dot{a}}{a}$	$\Gamma^\varphi_{\vartheta\varphi} = \frac{\sin\vartheta}{\cos\vartheta}$
$\Gamma^r{}_{tt} = 0$	$\Gamma^\varphi_{\vartheta\vartheta} = 0$
$\Gamma^r_{\vartheta\vartheta} = -r(1-Kr^2)$	$\Gamma^\varphi_{\varphi t} = \frac{\dot{a}}{a}$
$\Gamma^r_{\vartheta\varphi} = 0$	$\Gamma^\varphi_{t\varphi} = \frac{\dot{a}}{a}$
$\Gamma^r_{\varphi\vartheta} = 0$	$\Gamma^\varphi_{\varphi r} = \frac{1}{r}$
$\Gamma^r_{\varphi\varphi} = -r\sin^2\vartheta(1-Kr^2)$	$\Gamma^\varphi_{r\varphi} = \frac{1}{r}$
$\Gamma^i{}_{Aj} = 0$	$\Gamma^C_{ij} = 0$
	$\Gamma^C_{Ai} = 0$ if C $\neq$ A

A.5 Calculation Of Ricci Tensor And Ricci Scalar

According to (163), the off–diagonal terms of $g_{\mu\nu}$ are 0, and since by (161) the Γs and by (160) even the $R_{\mu\nu}$s contain only diagonal terms, we have two cases of Ricci tensors we have to calculate in order compose the Einstein tensor:

$$R_{ij} \quad \text{and} \quad R_{AB},$$

the R_{iA}s are zero.

By (160), we have

$$\begin{aligned}
R_{ij} = R^\alpha_{i\alpha j} &= \Gamma^\alpha_{i\alpha,j} - \Gamma^\alpha_{ij,\alpha} + \Gamma^\alpha_{\lambda j}\Gamma^\lambda_{i\alpha} - \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda{}_{ij} \\
&= \Gamma^k_{ik,j} + \Gamma^A_{iA,j} - \Gamma^k_{ij,k} - \Gamma^A_{ij,A} \\
&\quad + \Gamma^k{}_{lj}\Gamma^l{}_{ik} + \Gamma^k{}_{Aj}\Gamma^A_{ik} + \Gamma^A_{lj}\Gamma^l{}_{iA} + \Gamma^A{}_{Bj}\Gamma^B_{iA} \\
&\quad - \Gamma^k{}_{lk}\Gamma^l{}_{ij} - \Gamma^k{}_{Ak}\Gamma^A_{ij} + \Gamma^A_{lA}\Gamma^l{}_{ij} + \Gamma^A{}_{BA}\Gamma^B_{ij} \\
&= \underbrace{\Gamma^k_{ik,j} - \Gamma^k_{ij,k} + \Gamma^k{}_{lj}\Gamma^l{}_{ik} - \Gamma^k{}_{lk}\Gamma^l{}_{ij}}_{=\tilde{R}_{ij}\text{by (160)}} \\
&\quad + \Gamma^A_{iA,j} + \Gamma^A{}_{Bj}\Gamma^B_{iA} - \Gamma^A{}_{lA}\Gamma^l{}_{ij}.
\end{aligned} \tag{175}$$

In the same way, we get

$$\begin{aligned}
R_{AB} = R^\alpha_{A\alpha B} &= \widehat{R}_{AB} + \Gamma^k{}_{DB}\Gamma^D{}_{Ak} + \Gamma^E{}_{lB}\Gamma^l{}_{AE} \\
&\quad - \Gamma^k{}_{lk}\Gamma^l{}_{AB} - \Gamma^E{}_{lE}\Gamma^l{}_{AB} - \Gamma^k_{AB,k}.
\end{aligned}$$

We show the calculation of $R_{\vartheta\vartheta}$ with (176), the cases of R_{tt}, R_{rr} and $R_{\varphi\varphi}$ are analogous.

Writing out equation (176) for $A = B = \vartheta$:

$$\begin{aligned} R_{\vartheta\vartheta} = R^{\alpha}_{\vartheta\alpha\vartheta} &= \underbrace{\widehat{R}_{\vartheta\vartheta}}_{(a)} + \underbrace{\Gamma^{k}_{D\vartheta}\Gamma^{D}_{\vartheta k}}_{(b)} + \underbrace{\Gamma^{E}_{l\vartheta}\Gamma^{l}_{\vartheta E}}_{(c)} \\ &\quad - \underbrace{\Gamma^{k}{}_{lk}\Gamma^{l}_{\vartheta\vartheta}}_{(d)} - \underbrace{\Gamma^{E}{}_{lE}\Gamma^{l}_{\vartheta\vartheta}}_{(e)} - \underbrace{\Gamma^{k}_{\vartheta\vartheta,k}}_{(f)} . \end{aligned}$$

Every term can be calculated with the above table for the Γs.

(a)

$$\begin{aligned} R_{\vartheta\vartheta} &= R^{\alpha}_{\vartheta\alpha\vartheta} \\ &= \Gamma^{D}_{\vartheta D,\vartheta} - \Gamma^{D}_{\vartheta\vartheta,D} + \Gamma^{D}_{E\vartheta}\Gamma^{E}_{\vartheta D} - \Gamma^{D}_{ED}\Gamma^{E}_{\vartheta\vartheta} \\ &= \Gamma^{\vartheta}_{\vartheta\vartheta,\vartheta} + \Gamma^{\varphi}_{\vartheta\varphi,\vartheta} - \Gamma^{\vartheta}_{\vartheta\vartheta,\vartheta} - \Gamma^{\varphi}_{\varphi\vartheta,\varphi} \\ &\quad + \Gamma^{\vartheta}_{\vartheta\vartheta}\Gamma^{\vartheta}_{\vartheta\vartheta} + \Gamma^{\vartheta}_{\varphi\vartheta}\Gamma^{\varphi}_{\vartheta\vartheta} + \Gamma^{\varphi}_{\varphi\vartheta}\Gamma^{\varphi}_{\vartheta\varphi} + \Gamma^{\varphi}_{\vartheta\vartheta}\Gamma^{\vartheta}_{\vartheta\varphi} \\ &\quad - \Gamma^{\vartheta}_{\vartheta\vartheta}\Gamma^{\vartheta}_{\vartheta\vartheta} - \Gamma^{\vartheta}_{\varphi\vartheta}\Gamma^{\varphi}_{\vartheta\vartheta} - \Gamma^{\varphi}_{\vartheta\varphi}\Gamma^{\vartheta}_{\vartheta\vartheta} - \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\vartheta\vartheta} \\ &= \Gamma^{\varphi}_{\vartheta\varphi,\vartheta} + \Gamma^{\varphi}_{\varphi\vartheta}\Gamma^{\varphi}_{\vartheta\varphi} \end{aligned}$$

(b)

$$\Gamma^{k}_{D\vartheta}\Gamma^{D}_{\vartheta k} = \Gamma^{t}_{\vartheta\vartheta}\Gamma^{\vartheta}_{\vartheta t} + \underbrace{\Gamma^{t}_{\varphi\vartheta}\Gamma^{\varphi}_{\vartheta t}}_{0} + \Gamma^{r}_{\vartheta\vartheta}\Gamma^{\vartheta}_{\vartheta r} + \underbrace{\Gamma^{r}_{\varphi\vartheta}\Gamma^{\varphi}_{\vartheta r}}_{0}$$

(c)

$$\Gamma^{E}_{l\vartheta}\Gamma^{l}_{\vartheta E} = \Gamma^{\vartheta}_{t\vartheta}\Gamma^{t}_{\vartheta\vartheta} + \Gamma^{\vartheta}_{r\vartheta}\Gamma^{r}_{\vartheta\vartheta} + \underbrace{\Gamma^{\varphi}_{t\vartheta}\Gamma^{t}_{\vartheta\varphi}}_{0} + \underbrace{\Gamma^{\varphi}_{r\vartheta}\Gamma^{r}_{\vartheta\varphi}}_{0}$$

(d)

$$-\Gamma^{k}{}_{lk}\Gamma^{l}_{\vartheta\vartheta} = -\underbrace{\Gamma^{t}{}_{tt}\Gamma^{t}_{\vartheta\vartheta}}_{0} - \underbrace{\Gamma^{t}{}_{rt}\Gamma^{r}_{\vartheta\vartheta}}_{0} - \Gamma^{r}{}_{tr}\Gamma^{t}_{\vartheta\vartheta} - \Gamma^{r}{}_{rr}\Gamma^{r}_{\vartheta\vartheta}$$

(e)

$$\begin{aligned} -\Gamma^{E}{}_{lE}\Gamma^{l}_{\vartheta\vartheta} &= -\Gamma^{\vartheta}_{t\vartheta}\Gamma^{t}_{\vartheta\vartheta} - \Gamma^{\vartheta}_{r\vartheta}\Gamma^{r}_{\vartheta\vartheta} - \Gamma^{\varphi}_{t\varphi}\Gamma^{t}_{\vartheta\vartheta} - \Gamma^{\varphi}_{r\varphi}\Gamma^{r}_{\vartheta\vartheta} \\ &= -2\Gamma^{\vartheta}_{t\vartheta}\Gamma^{t}_{\vartheta\vartheta} - 2\Gamma^{\vartheta}_{r\vartheta}\Gamma^{r}_{\vartheta\vartheta} \end{aligned}$$

(f)

$$-\Gamma^{k}_{\vartheta\vartheta,k} = -\Gamma^{t}_{\vartheta\vartheta,t} - \Gamma^{r}_{\vartheta\vartheta,r}$$

Adding all parts of $R_{\vartheta\vartheta}$ from (a) to (f) gives

$$
\begin{aligned}
R_{\vartheta\vartheta} &= \Gamma^{\varphi}_{\vartheta\varphi,\vartheta} + \Gamma^{\varphi}_{\varphi\vartheta}\Gamma^{\varphi}_{\vartheta\varphi} + \Gamma^{t}_{\vartheta\vartheta}\Gamma^{\vartheta}_{\vartheta t} + \Gamma^{\vartheta}_{t\vartheta}\Gamma^{t}_{\vartheta\vartheta} + \Gamma^{\vartheta}_{r\vartheta}\Gamma^{r}_{\vartheta\vartheta} \\
&\quad -\Gamma^{r}{}_{tr}\Gamma^{t}_{\vartheta\vartheta} - \Gamma^{r}{}_{rr}\Gamma^{r}_{\vartheta\vartheta} - 2\Gamma^{\vartheta}_{t\vartheta}\Gamma^{t}_{\vartheta\vartheta} - 2\Gamma^{\vartheta}_{r\vartheta}\Gamma^{r}_{\vartheta\vartheta} - \Gamma^{t}_{\vartheta\vartheta,t} - \Gamma^{r}_{\vartheta\vartheta,r} \\
&= \Gamma^{\varphi}_{\vartheta\varphi,\vartheta} + \Gamma^{\varphi}_{\varphi\vartheta}\Gamma^{\varphi}_{\vartheta\varphi} + \underbrace{\Gamma^{t}_{\vartheta\vartheta}\Gamma^{\vartheta}_{\vartheta t} - \Gamma^{\vartheta}_{t\vartheta}\Gamma^{t}_{\vartheta\vartheta}}_{0} - \Gamma^{\vartheta}_{r\vartheta}\Gamma^{r}_{\vartheta\vartheta} \\
&\quad -\Gamma^{r}{}_{tr}\Gamma^{t}_{\vartheta\vartheta} - \Gamma^{r}{}_{rr}\Gamma^{r}_{\vartheta\vartheta} - \Gamma^{t}_{\vartheta\vartheta,t} - \Gamma^{r}_{\vartheta\vartheta,r} \\
&= \frac{\partial}{\partial\vartheta}\left(\frac{\cos\vartheta}{\sin\vartheta}\right) + \frac{\cos\vartheta^2}{\sin^2\vartheta} + \frac{1}{r}r\left(1-Kr^2\right) \\
&\quad -\frac{\dot a}{a}a\dot a r^2 - \frac{\partial}{\partial t}\left(a\dot a r^2\right) - \frac{\partial}{\partial r}\left(-r(1-Kr^2)\right) \\
&= -r^2\left(2\dot a^2 + a\ddot a + 2K\right).
\end{aligned}
$$

In the same fashion we find the other $R_{\mu\nu}$s. The result is

$$
\begin{aligned}
R_{tt} &= 3\frac{\ddot a}{a} \\
R_{rr} &= -\frac{1}{1-Kr^2}\left(2\dot a^2 + a\ddot a + 2K\right) \\
R_{\vartheta\vartheta} &= -r^2\left(2\dot a^2 + a\ddot a + 2K\right) \\
R_{\varphi\varphi} &= -r^2\sin^2\vartheta\left(2\dot a^2 + a\ddot a + 2K\right).
\end{aligned}
$$

The Ricci Scalar is built from these quantities:

$$
\begin{aligned}
R = R^{\lambda}{}_{\lambda} &= g^{\lambda\sigma}R_{\sigma\lambda} \qquad (176) \\
&= g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\vartheta\vartheta}R_{\vartheta\vartheta} + g^{\varphi\varphi}R_{\varphi\varphi} \\
&= -3\frac{\ddot a}{a} - \frac{1-Kr^2}{a^2}\frac{a\ddot a + 2\dot a^2 + 2K}{1-Kr^2} - \frac{1}{a^2r^2}r^2\left(a\ddot a + 2\dot a^2 + 2K\right) \\
&\quad -\frac{1}{r^2\sin^2\vartheta}r^2\sin^2\vartheta\left(a\ddot a + 2\dot a^2 + 2K\right) \\
&= -6\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{K}{a^2}\right).
\end{aligned}
$$

A.6 Calculation Of Einstein Tensor

Writing out (158), the aim to calculate the left side of Einstein's equations (156) means to calculate

$$
\begin{aligned}
G_{tt} &= R_{tt} - \frac{1}{2}g_{tt}R \\
G_{rr} &= R_{rr} - \frac{1}{2}g_{rr}R \\
G_{\vartheta\vartheta} &= R_{\vartheta\vartheta} - \frac{1}{2}g_{\vartheta\vartheta}R \\
G_{\varphi\varphi} &= R_{\varphi\varphi} - \frac{1}{2}g_{\varphi\varphi}R
\end{aligned}
$$

with the $R_{\mu\nu}$s and R listed at the end of the previous section. Thus, we get

$$\begin{aligned}
G_{tt} &= R_{tt} - \frac{1}{2} g_{tt} R \\
&= 3\frac{\ddot{a}}{a} - \frac{1}{2}\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + 6\frac{K}{a^2}\right) \\
G_{rr} &= R_{rr} - \frac{1}{2} g_{rr} R \\
&= -\frac{\left(2\dot{a}^2 + a\ddot{a} + 2K\right)}{1 - Kr^2} + \frac{1}{2}\frac{a^2}{1 - Kr^2}\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + 6\frac{K}{a^2}\right) \\
G_{\vartheta\vartheta} &= R_{\vartheta\vartheta} - \frac{1}{2} g_{\vartheta\vartheta} R \\
&= -r^2\left(2\dot{a}^2 + a\ddot{a} + 2K\right) + \frac{1}{2}a^2 r^2\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + 6\frac{K}{a^2}\right) \\
G_{\varphi\varphi} &= R_{\varphi\varphi} - \frac{1}{2} g_{\varphi\varphi} R \\
&= -r^2 \sin^2\vartheta\left(2\dot{a}^2 + a\ddot{a} + 2K\right) + \frac{1}{2}a^2 r^2 \sin^2\vartheta\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + 6\frac{K}{a^2}\right),
\end{aligned}$$

and this leads to the result

$$G_{tt} = -3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right) \tag{177}$$

$$G_{rr} = \frac{1}{1 - Kr^2}\left(2\ddot{a}a + \dot{a}^2 + K\right) \tag{178}$$

$$G_{\vartheta\vartheta} = r^2\left(2\ddot{a}a + \dot{a}^2 + K\right) \tag{179}$$

$$G_{\varphi\varphi} = r^2 \sin^2\vartheta\left(2\ddot{a}a + \dot{a}^2 + K\right). \tag{180}$$

But this is not Einstein's equations yet, it is only the Einstein tensor. We still have to calculate the right side of (156). Fortunately, this is rather easy compared to calculating the left side.

A.7 Einstein's Equations Completed

Taking for granted that

$$T_{\mu\nu} = \rho u_\mu u_\nu + P\left(\eta_{\mu\nu} + u_\mu u_\nu\right) \tag{181}$$

is the *stress-energy tensor* describing a continuous distribution of matter called a *perfect fluid*[27], we start by remarking that this tensor is written with the *Minkowski metric*

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{182}$$

[27] [6], chapter 4.2.

which is different from the Robertson-Walker metric (159) we were using to calculate the left side of Einstein's equations. At first glance, this means we have to transform either side of Einstein's equations to find the right expressions in the same metric.

But raising an index on both sides of Einstein's equation makes it independent of the metric used, since all metric-dependend factors in either $G_{\mu\nu}$ or $T_{\mu\nu}$ divide out:

$$\begin{aligned} G_{\mu\nu} &= 8\pi T_{\mu\nu} \\ \Leftrightarrow g^{\mu\lambda} G_{\lambda\nu} &= 8\pi g^{\mu\lambda} \underbrace{T_{\lambda\nu}}_{g_{\mu\nu}} = 8\pi \eta^{\mu\lambda} \underbrace{T_{\lambda\nu}}_{\eta_{\mu\nu}} \\ \Leftrightarrow G^{\mu}{}_{\nu} &= 8\pi T^{\mu}{}_{\nu} \end{aligned} \tag{183}$$

This means we can calculate the left side in the Robertson-Walker metric and the right side with the Minkowski metric and afterwards raise the first index to render the result metric-independent.

Since our results are coordinate-independent, we can calculate $T_{\mu\nu}$ in the inertial system at rest, where the velocity vector is

$$u^{\mu} = (1, 0, 0, 0) \tag{184}$$

in components. Thus, according to (181), (182) and (184),

$$\begin{aligned} T_{00} &= \rho u_0 u_0 + P(\eta_{00} + u_0 u_0) \\ &= \rho + P(-1 + 1) = \rho \\ T_{11} &= \rho u_1 u_1 + P(\eta_{11} + u_1 u_1) \\ &= P(1 + 0) = P = T_{22} = T_{33}, \end{aligned}$$

where we used the indices $(0, 1, 2, 3)$, to distinguish results in the $\eta_{\mu\nu}$ metric from results in the Robertson-Walker metric. The stress-energy tensor for a perfect fluid looks like

$$\mathbf{T}_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}. \tag{185}$$

We now see the metric-independent result for Einstein's equations with (183), (177), (164) and the above values for $T_{\mu\nu}$:

$$\begin{aligned} g^{tt} G_{tt} = G^{t}{}_{t} = 3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right) &= -8\pi\rho = T^{0}_{\ 0} = \eta^{00} T_{00} \\ g^{rr} G_{rr} = G^{r}{}_{r} = 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} &= 8\pi P = T^{1}_{\ 1} = \eta^{11} T_{11} \end{aligned}$$

$$
\begin{aligned}
G^{\vartheta}{}_{\vartheta} = 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} &= 8\pi P = T^2{}_2 \\
G^{\varphi}{}_{\varphi} = 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} &= 8\pi P = T^3{}_3.
\end{aligned}
$$

B Notation

- We always use Einstein's summation convention for tensors: $\sum_{k=1}^{n} A^{ijk}_{\ \ \ kl} := A^{ijk}_{\ \ \ kl}$, and $\sum_{k=1}^{n} B^{k}_{k} := B^{k}_{k}$
- We write $\frac{\partial}{\partial x^i}$ simply as $_{,i}$.
- Latin letters refer to conceptual ideas of tensors while greek letters indicate components. Often, latin and greek letters can be used exchangeably, but often it makes things a lot easier to see.
- We mark derivations by time with a dot: $\frac{\partial a}{\partial t} := \dot{a}$.
- Quantities and objects that live on the tangent spaces of the Σs defined in (1) are marked with a bar, like $\bar{\beta}$ or $\bar{e}_i$.
- The Kronecker delta is used very often:

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

  Note that $\delta^i_{\ i}$ means to sum over all i, often the sum over the three space indices from the 3+1 split. In this context, $\delta^i_{\ i} = 3$.
- We denote $\nabla^i \nabla^j$ by $^{|ij}$ and $\nabla_i \nabla_j$ by $_{|ij}$.
- δ without indices denotes perturbed quantities, e.g. δg is the perturbed metric corresponding to g.

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